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Handout 28 Principles of Digital Communications

Solutions to Problem Set 12 May 26, 2015

SOLUTION 1.

(a) Let $g_{\mathcal{F}}(f) = \mathbb{1}\{-\frac{B}{2} \le f \le \frac{B}{2}\}$. Its inverse Fourier transform is $g(t) = B\,\mathrm{sinc}(Bt)$. We note that $p_{\mathcal{F}}(f) = g_{\mathcal{F}}(f+f_0) + g_{\mathcal{F}}(f-f_0)$, therefore

$$\begin{aligned} p(t) &= g(t)\left(e^{-\mathrm{j}2\pi f_0 t} + e^{\mathrm{j}2\pi f_0 t}\right) \\ &= 2B\,\mathrm{sinc}(Bt)\cos(2\pi f_0 t). \end{aligned}$$

(b) We have

$$\begin{aligned} \int_{-\infty}^{\infty} \psi^2(t)dt &= c^2 \int_{-\infty}^{\infty} p^2(t)\ dt \\ &= c^2 \int_{-\infty}^{\infty} p_{\mathcal{F}}^2(f)\ df \\ &= 2c^2 B = 1, \end{aligned}$$

which gives $c = \frac{1}{\sqrt{2B}}$.

(c) $\{\psi(t-nT)\}_{n\in\mathbb{Z}}$ forms an orthonormal set if and only if Nyquist's criterion with parameter T holds. In other words,

$$\sum_{n=-\infty}^{\infty} \left|\psi_{\mathcal{F}}\left(f - \frac{n}{T}\right)\right|^2 = T.$$

A simple drawing of the frequency domain for $T = \frac{1}{2B}$ shows that Nyquist's criterion holds. Therefore $\{\psi(t-nT)\}_{n\in\mathbb{Z}}$ forms an orthonormal set.

(d) In applying Nyquist's criterion, we picture the periodic repetition (with period $\frac{1}{T}$) of the positive frequencies of $|p_{\mathcal{F}}(f)|^2$ as filling half of the real line and the periodic repetition of the negative frequencies as filling the remaining gaps. This happens if and only if, when we shift the frequency interval $[f_0 - \frac{B}{2}, f_0 + \frac{B}{2}]$ by some multiple of $\frac{1}{T} = 2B$, the left end of the shifted interval corresponds to the right end of $[-f_0 - \frac{B}{2}, -f_0 + \frac{B}{2}]$, i.e., if $-f_0 + \frac{B}{2} + k2B = f_0 - \frac{B}{2}$. Solving the former gives $f_0 = \frac{B}{2} + kB$, where k is an integer.

SOLUTION 2.

(a) Let $x_E(t) = x_R(t) + \mathrm{j}x_I(t)$. Then

$$\begin{aligned} x(t) &= \sqrt{2}\Re\{x_E(t)e^{\mathrm{j}2\pi f_c t}\} \\ &= \sqrt{2}\Re\{[x_R(t) + \mathrm{j}x_I(t)]e^{\mathrm{j}2\pi f_c t}\} \\ &= \sqrt{2}[x_R(t)\cos(2\pi f_c t) - x_I(t)\sin(2\pi f_c t)]. \end{aligned}$$

Hence, we have

$$x_{EI}(t) = \sqrt{2}\Re\{x_E(t)\}$$

and

$$x_{EQ}(t) = \sqrt{2}\Im\{x_E(t)\}.$$

(b) Let $x_E(t) = \alpha(t)e^{\mathrm{j}\beta(t)}$. Then

$$\begin{aligned} x(t) &= \sqrt{2}\Re\{x_E(t)e^{\mathrm{j}2\pi f_c t}\} \\ &= \sqrt{2}\Re\{\alpha(t)e^{\mathrm{j}\beta(t)}e^{\mathrm{j}2\pi f_c t}\} \\ &= \sqrt{2}\Re\{\alpha(t)e^{\mathrm{j}(2\pi f_c t+\beta(t))}\} \\ &= \sqrt{2}\alpha(t)\cos[2\pi f_c t + \beta(t)]. \end{aligned}$$

We thus have

$$x_E(t) = \alpha(t)e^{\mathrm{j}\beta(t)} = \frac{a(t)}{\sqrt{2}}e^{\mathrm{j}\theta(t)}.$$

(c) From (b) we see that

$$x_E(t) = \frac{A(t)}{\sqrt{2}}e^{\mathrm{j}\varphi}.$$

This is consistent with Example 7.9 (DSB-SC) given in the text. We can also verify:

$$\begin{aligned} x(t) &= \sqrt{2}\Re\{x_E(t)e^{\mathrm{j}2\pi f_c t}\} \\ &= \sqrt{2}\Re\left\{\frac{A(t)}{\sqrt{2}}e^{\mathrm{j}\varphi}e^{\mathrm{j}2\pi f_c t}\right\} \\ &= \Re\{A(t)e^{\mathrm{j}(2\pi f_c t+\varphi)}\} \\ &= A(t)\cos(2\pi f_c t + \varphi). \end{aligned}$$

SOLUTION 3.

(a) The key observation is that while $e^{\mathrm{j}2\pi f_1 t}$ and $e^{-\mathrm{j}2\pi f_1 t}$ are two different signals if $f_1 \neq 0$, $\Re\{e^{\mathrm{j}2\pi f_1 t}\}$ and $\Re\{e^{-\mathrm{j}2\pi f_1 t}\}$ are identical.

Therefore, if we fix $f_1 \neq 0$ and choose $a_1(t)$ and $a_2(t)$ so that $a_1(t)e^{\mathrm{j}2\pi f_c t} = e^{\mathrm{j}2\pi f_1 t}$ and $a_2(t)e^{\mathrm{j}2\pi f_c t} = e^{-\mathrm{j}2\pi f_1 t}$, we get $a_1(t) \neq a_2(t)$ and $\Re\left\{a_1(t)e^{\mathrm{j}2\pi f_c t}\right\} = \Re\left\{a_2(t)e^{\mathrm{j}2\pi f_c t}\right\}$.

Let $a_1(t) = e^{-\mathrm{j}2\pi(f_c-f_1)t}$ and $a_2(t) = e^{-\mathrm{j}2\pi(f_c+f_1)t}$. Then $a_1(t) \neq a_2(t)$ and

$$\sqrt{2}\Re\left\{a_1(t)e^{\mathrm{j}2\pi f_c t}\right\} = \sqrt{2}\Re\left\{a_2(t)e^{\mathrm{j}2\pi f_c t}\right\}.$$

(b) Let $b(t) = a(t)e^{\mathrm{j}2\pi f_c t}$, which represents a translation of $a(t)$ in the frequency domain: If $a_{\mathcal{F}}(f) = 0$ for $f < -f_c$, then $b_{\mathcal{F}}(f) = 0$ for $f < 0$. Because $\Re\{b(t)\} = \frac{1}{2}\left(a(t)e^{\mathrm{j}2\pi f_c t} + a^*(t)e^{-\mathrm{j}2\pi f_c t}\right)$, taking the real part has a scaling effect and adds a negative-frequency component. The negative spectrum is canceled by the $h_>$ filter, and the scaling is compensated by the $\sqrt{2}$ factors from the up-converter and down-converter. Multiplying by $e^{-\mathrm{j}2\pi f_c t}$ translates the spectrum back to the initial position. In conclusion, we obtain $a(t)$.

(c) Take any baseband signal $u(t)$ with frequency domain support $[-fc - \Delta, fc + \Delta]$, $\Delta > 0$. The signal can be real-valued or complex-valued (for example $u_{\mathcal{F}}(f) = \mathbb{1}_{[-fc-\Delta, fc+\Delta]}(f)$, which is a sinc in time domain). After we up-convert, the support of $u_{\mathcal{F}}(f)$ will not extend beyond $2f_c + \Delta$. When we chop the negative frequencies we obtain a support contained in $[0, 2f_c + \Delta]$ and when we shift back to the left the support will be contained in $[-f_c, f_c + \Delta]$, which is too small to be the support of $u_{\mathcal{F}}(f)$.

(d) In time domain:

$$\begin{aligned} w(t) &= \sqrt{2}\Re\{a(t)e^{\mathrm{j}2\pi f_c t}\} \\ &\overset{a\in\mathbb{R}}{=} \sqrt{2}a(t)\cos(2\pi f_c t). \end{aligned}$$

Therefore,

$$a(t) = \frac{w(t)}{\sqrt{2}\cos(2\pi f_c t)}.$$

In frequency domain: If $a_{\mathcal{F}}(f) = 0$ for $f < -f_c$, we obtain $a(t)$ as described in (b). In the following, we consider the case $a_{\mathcal{F}}(f) \neq 0$ for $f < -f_c$.

We have $w_{\mathcal{F}}(f) = \frac{1}{\sqrt{2}}\left[a_{\mathcal{F}}(f - f_c) + a_{\mathcal{F}}(f + f_c)\right] = a^+_{\mathcal{F}}(f) + a^-_{\mathcal{F}}(f)$, with $a^+_{\mathcal{F}}(f) = \frac{1}{\sqrt{2}}a_{\mathcal{F}}(f - f_c)$ and $a^-_{\mathcal{F}}(f) = \frac{1}{\sqrt{2}}a_{\mathcal{F}}(f + f_c)$, respectively. These two components have overlapping support in some interval centered at 0. However, there is no overlap for sufficiently large frequencies. This means that for sufficiently large frequencies f we have $w_{\mathcal{F}}(f) = \frac{1}{\sqrt{2}}a^+_{\mathcal{F}}(f)$, which implies that from $w_{\mathcal{F}}(f)$ we can observe the right tail of $a^+_{\mathcal{F}}(f)$ and use that information to remove the right tail of $a^-_{\mathcal{F}}(f)$ from $w_{\mathcal{F}}(f)$ (the right tails of $a^+_{\mathcal{F}}(f)$ and $a^-_{\mathcal{F}}(f)$ are the same because $a(t)$ is real). Hence, from $w_{\mathcal{F}}(f)$ we can read more of the right tail of $a^+_{\mathcal{F}}(f)$. The procedure can be repeated until we get to see $a^+_{\mathcal{F}}(f)$ for all frequencies above f_c. At this point, using $a_{\mathcal{F}}(f) = a^+_{\mathcal{F}}(f + f_c)\sqrt{2}$ and the fact that $a(t)$ is real-valued, we have $a_{\mathcal{F}}(f)$ for the positive frequencies, hence for all frequencies.

Solution 4.

$$\begin{aligned}
x(t)\sqrt{2}\cos(2\pi f_c t) &= x(t)\left[\frac{e^{\mathrm{j}2\pi f_c t}+e^{-\mathrm{j}2\pi f_c t}}{\sqrt{2}}\right] \\
&= \sqrt{2}\Re\{x_E(t)e^{\mathrm{j}2\pi f_c t}\}\left[\frac{e^{\mathrm{j}2\pi f_c t}+e^{-\mathrm{j}2\pi f_c t}}{\sqrt{2}}\right] \\
&= \left[\frac{x_E(t)e^{\mathrm{j}2\pi f_c t}+x_E^*(t)e^{-\mathrm{j}2\pi f_c t}}{\sqrt{2}}\right]\left[\frac{e^{\mathrm{j}2\pi f_c t}+e^{-\mathrm{j}2\pi f_c t}}{\sqrt{2}}\right] \\
&= \frac{x_E(t)e^{\mathrm{j}4\pi f_c t}+x_E(t)+x_E^*(t)+x_E^*(t)e^{-\mathrm{j}4\pi f_c t}}{2}.
\end{aligned}$$

At the lowpass filter output we have

$$\frac{x_E(t)+x_E^*(t)}{2}=\Re\{x_E(t)\}.$$

The calculation for the other path is similar.

Solution 5.

(a) Notice that the sinusoids of $w(t)$ have a period of $T_c = 4$ ms units of time, which implies that $f_c = \frac{1}{T_c} = \frac{1}{4\text{ ms}} = 250$ Hz.

(b) Notice that the phase of the sinusoidal signal changes every $T_s = 4$ ms. (Here we have $T_s = T_c$, but in general it is not the case. In practice we usually have $T_s \gg T_c$. See the note at the end.)

The expression of $w(t)$ as a function of t is:

$$\begin{aligned}
w(t) &= \begin{cases} 4\cos(2\pi f_c t - \frac{\pi}{2}) & t \in]0, T_s[\\ 4\cos(2\pi f_c t) & t \in]T_s, 2T_s[\\ 4\cos(2\pi f_c t + \pi) & t \in]2T_s, 3T_s[\\ 4\cos(2\pi f_c t + \frac{\pi}{2}) & t \in]3T_s, 4T_s[\end{cases} = \begin{cases} \Re\left\{4e^{j(2\pi f_c t - \frac{\pi}{2})}\right\} & t \in]0, T_s[\\ \Re\left\{4e^{j(2\pi f_c t)}\right\} & t \in]T_s, 2T_s[\\ \Re\left\{4e^{j(2\pi f_c t + \pi)}\right\} & t \in]2T_s, 3T_s[\\ \Re\left\{4e^{j(2\pi f_c t + \frac{\pi}{2})}\right\} & t \in]3T_s, 4T_s[\end{cases} \\
&= \begin{cases} \Re\left\{-4je^{j2\pi f_c t}\right\} & t \in]0, T_s[\\ \Re\left\{4e^{j2\pi f_c t}\right\} & t \in]T_s, 2T_s[\\ \Re\left\{-4e^{j2\pi f_c t}\right\} & t \in]2T_s, 3T_s[\\ \Re\left\{4je^{j2\pi f_c t}\right\} & t \in]3T_s, 4T_s[\end{cases} = \sqrt{2}\Re\left\{w_E(t)e^{j2\pi f_c t}\right\},
\end{aligned}$$

where

$$\begin{aligned} w_E(t) =& -\frac{4j}{\sqrt{2}}\mathbb{1}\{t\in]0,T_s[\} + \frac{4}{\sqrt{2}}\mathbb{1}\{t\in]T_s,2T_s[\} \\ & -\frac{4}{\sqrt{2}}\mathbb{1}\{t\in]2T_s,3T_s[\} + \frac{4j}{\sqrt{2}}\mathbb{1}\{t\in]3T_s,4T_s[\} \\ =& -j\sqrt{8T_s}\frac{1}{\sqrt{T_s}}\mathbb{1}\{t\in]0,T_s[\} + \sqrt{8T_s}\frac{1}{\sqrt{T_s}}\mathbb{1}\{t\in]T_s,2T_s[\} \\ & -\sqrt{8T_s}\frac{1}{\sqrt{T_s}}\mathbb{1}\{t\in]2T_s,3T_s[\} + j\sqrt{8T_s}\frac{1}{\sqrt{T_s}}\mathbb{1}\{t\in]3T_s,4T_s[\}. \end{aligned}$$

If we define $\psi(t) = \frac{1}{\sqrt{T_s}}\mathbb{1}\{t\in]0,T_s[\}$, $c_0 = -j\sqrt{8T_s}$, $c_1 = \sqrt{8T_s}$, $c_2 = -\sqrt{8T_s}$ and $c_3 = j\sqrt{8T_s}$, we get

$$w_E(t) = \sum_{i=0}^{3} c_i\psi(t - iT_s). \tag{1}$$

Therefore, the pulse used in the waveform former is $\psi(t) = \frac{1}{\sqrt{T_s}}\mathbb{1}\{t\in]0,T_s[\}$, and the waveform former output signal is given by (1). The orthonormal basis that is used is $\{\psi(t-iT_s)\}_{i=0}^{3}$.

(c) The symbol sequence is $\{c_0,\ c_1,\ c_2,\ c_3\} = \left\{-j\sqrt{\mathcal{E}_s},\ \sqrt{\mathcal{E}_s},\ -\sqrt{\mathcal{E}_s},\ j\sqrt{\mathcal{E}_s}\right\}$, where $\mathcal{E}_s = 8T_s$. We can see that the symbol alphabet is $\left\{\sqrt{\mathcal{E}_s},\ j\sqrt{\mathcal{E}_s},\ -\sqrt{\mathcal{E}_s},\ -j\sqrt{\mathcal{E}_s}\right\}$.

(d) We have:

- The output sequence of the encoder is the symbol sequence, which is

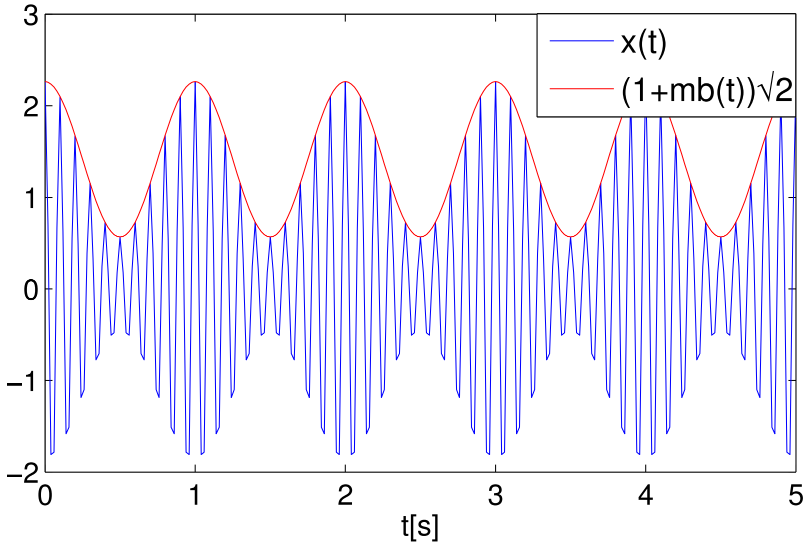
$$\{c_0,\ c_1,\ c_2,\ c_3\} = \left\{-j\sqrt{\mathcal{E}_s},\ \sqrt{\mathcal{E}_s},\ -\sqrt{\mathcal{E}_s},\ j\sqrt{\mathcal{E}_s}\right\}.$$

- The symbol alphabet contains 4 symbols. This means that each symbol represents two bits. Now since the symbol rate is $f_s = \frac{1}{T_s} = 250$ symbols/s, the bit rate is $2 \times 250 = 500$ bits/s.
- The input/output mapping can be obtained by assigning two bits for each symbol in the symbol alphabet. Keeping in mind that it is better to minimize the number of bit-differences between close symbols, we obtain the following input/output mapping (which is not unique, i.e., we can obtain other mappings that satisfy the mentioned criterion): $\sqrt{\mathcal{E}_s} \longleftrightarrow 00$, $j\sqrt{\mathcal{E}_s} \longleftrightarrow 01$, $-\sqrt{\mathcal{E}_s} \longleftrightarrow 11$ and $-j\sqrt{\mathcal{E}_s} \longleftrightarrow 10$.
- Assuming that the above input/output mapping was used, we can obtain the input sequence of the encoder: 10001101.

Note that in this example, we have $T_s = T_c$, so $f_c = f_s$. This is very unusual. In practice we almost always have $f_c \gg f_s$, especially if we are using electromagnetic waves.

Solution 6.

(a) $x(t)$ is a sinusoid with instantaneous amplitude $(1+mb(t))\sqrt{2}$, therefore it will always be in the interval $\sqrt{2}[-1-mb(t), 1+mb(t)]$. In conclusion, the envelope of $|x(t)|$ will be $(1+mb(t))\sqrt{2}$. Below, we show an example for $f_c = 10$ Hz, $b(t) = \cos(2\pi t)$ and $m = 0.6$.



(b)

$$
\begin{aligned}
|x(t)| &= (1+mb(t))\sqrt{2}|\cos(2\pi f_c t)| \\
&\overset{(*)}{=} (1+mb(t))\sqrt{2}\sum_{k=-\infty}^{\infty} c_k e^{\mathrm{j}2\pi(2kf_c)t},
\end{aligned}
$$

where $(*)$ is obtained by expanding the periodic signal $|\cos(2\pi f_c t)|$ of period $T = \frac{1}{2f_c}$ as a Fourier series. Note that each term of the series has frequency $2kf_c$. Therefore, if we pass $|x(t)|$ through an ideal lowpass filter with the cutoff frequency f_0 in the interval $[B, 2f_c - B]$, we will only keep the central term and obtain $1+mb(t)$, scaled by $c_0\sqrt{2}$.

(c) The diode and the parallel R_1C_1 circuit form an envelope detector, so the voltage on top of R_1 is $(1+mb(t))\sqrt{2}$. If we read the voltage on top of R_2, the series R_2C_2 circuit is a highpass filter, which eliminates the constant component, and we obtain a scaled version of $b(t)$. (We assume that $b(t)$ does not contain low frequencies which will be affected by the filter.)