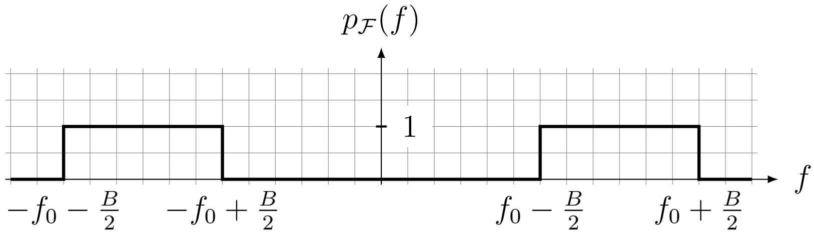
ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 27 Principles of Digital Communications
Problem Set 12 May 20, 2015

PROBLEM 1. *(Bandpass Pulses)* Let $p(t)$ be the pulse that has Fourier transform as shown below:



(a) Determine $p(t)$.

(b) Determine the constant c so that $\psi(t) = cp(t)$ has unit energy.

(c) Assume that $f_0 - \frac{B}{2} = B$ and consider the infinite set of functions $\{\psi(t - lT)\}_{l \in \mathbb{Z}}$. Do they form an orthonormal set for $T = \frac{1}{2B}$?

(d) Determine all possible values of f_0 so that $\{\psi(t - lT)\}_{l \in \mathbb{Z}}$ forms an orthonormal set for $T = \frac{1}{2B}$.

PROBLEM 2. *(Equivalent Representations)* A real-valued passband signal $x(t)$ can be written as $x(t) = \sqrt{2}\Re\{x_E(t)e^{j2\pi f_c t}\}$, where $x_E(t)$ is the baseband-equivalent signal (complex-valued in general) with respect to the carrier frequency f_c. Also, a general complex-valued signal $x_E(t)$ can be written in terms of two real-valued signals, either as $x_E(t) = u(t) + jv(t)$ or as $\alpha(t)\exp\{j\beta(t)\}$.

(a) Show that a real-valued passband signal $x(t)$ can always be written as

$$x_{EI}(t)\cos(2\pi f_c t) - x_{EQ}(t)\sin(2\pi f_c t)$$

and relate $x_{EI}(t)$ and $x_{EQ}(t)$ to $x_E(t)$. Note: This formula can be used at the sender to produce $x(t)$ without doing complex-valued operations. The signals $x_{EI}(t)$ and $x_{EQ}(t)$ are called the *in-phase* and the *quadrature* components, respectively.

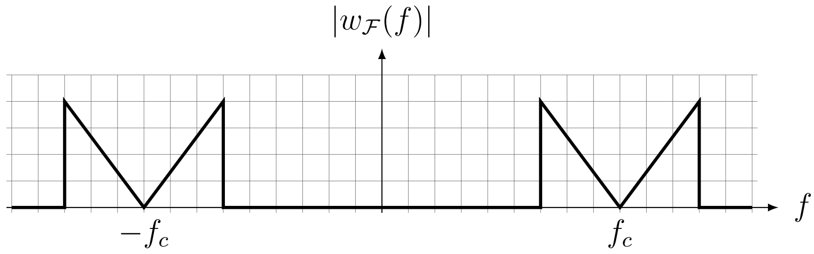
(b) Show that a real-valued passband signal $x(t)$ can always be written as

$$a(t)\cos[\pi f_c t + \theta(t)]$$

and relate $x_E(t)$ to $a(t)$ and $\theta(t)$. Note: This explains why sometimes people make the claim that a passband signal is modulated in amplitude and in phase.

(c) Use Part (b) to find the baseband-equivalent of the signal $x(t) = A(t)\cos(2\pi f_c t + \varphi)$, where $A(t)$ is a real-valued lowpass signal. Verify your answer with Example 7.9 where we assumed $\varphi = 0$.

PROBLEM 3. *(Passband)* Let f_c be a positive carrier frequency and consider an arbitrary real-valued function $w(t)$. You can visualize its Fourier transform as shown below:



(a) Argue that there are two different functions, $a_1(t)$ and $a_2(t)$, such that, for $i = 1, 2$,

$$w(t) = \sqrt{2}\Re\{a_i(t)\exp(\mathrm{j}2\pi f_c t)\}.$$

This shows that, without some constraint on the input signal, the operation performed by the circuit of Figure 7.4(b) is not reversible, even in the absence of noise. This was already pointed out in the discussion preceding Lemma 7.8.

(b) Argue that if we limit the input of Figure 7.4(b) to signals $a(t)$ such that $a_{\mathcal{F}}(f) = 0$ for $f < -f_c$, then the circuit of Figure 7.4(a) will retrieve $a(t)$ when fed with the output of Figure 7.4(b).

(c) Find an example showing that the condition of Part (b) is necessary. (Can you find an example with a real-valued $a(t)$?)

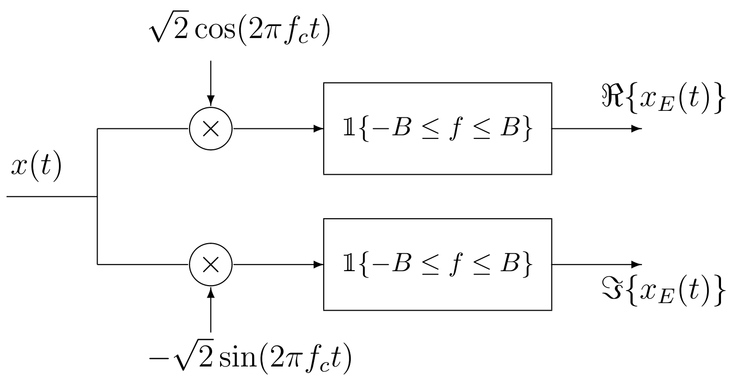
(d) Argue that if we limit the input of Figure 7.4(b) to signals $a(t)$ that are real-valued, then the input of Figure 7.4(b) can be retrieved from the output.

Hint 1: We are not claiming that the circuit of Figure 7.4(a) will retrieve $a(t)$.

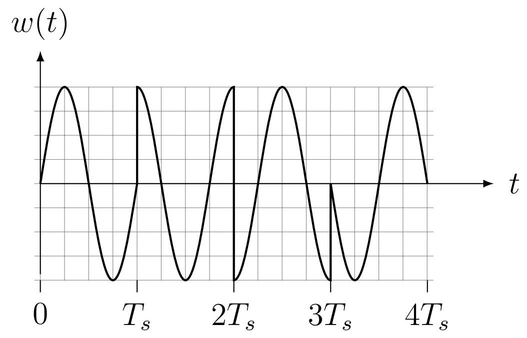
Hint 2: You may argue in the time domain or in the frequency domain. If you argue in the time domain, you can assume that $a(t)$ is continuous. In the frequency domain argument, you can assume that $a(t)$ has finite bandwidth.

PROBLEM 4. *(From Passband to Baseband via Real-Valued Operations)* Let the signal $x_E(t)$ be bandlimited to $[-B, B]$ and let $x(t) = \sqrt{2}\Re\{x_E(t)e^{\mathrm{j}2\pi f_c t}\}$, where $0 < B < f_c$. Show that the following circuit, when fed with $x(t)$, recovers the real and imaginary part of $x_E(t)$. (The two boxes are ideal lowpass filters of cutoff frequency B.)

Note: the circuit uses only real-valued operations.



PROBLEM 5. *(Reverse Engineering)* A toy passband signal is shown below (its carrier frequency is unusually small with respect to the symbol time T_s).



Suppose that the horizontal time scale is 1 ms per square and the vertical scale is 1 unit per square. Specify the three layers of a transmitter that generates the given signal, namely:

(a) The carrier frequency f_c used by the up-converter.

(b) The orthonormal basis used by the waveform former to produce the baseband-equivalent signal $w_E(t)$.

(c) The symbol alphabet, seen as a subset of $\mathbb{C}$.

(d) An encoding map, the encoder input sequence that leads to $w(t)$, the bit rate, the encoder output sequence, and the symbol rate.

PROBLEM 6. *(AM Receiver)* Let $x(t) = (1 + mb(t))\sqrt{2}\cos(2\pi f_c t)$ be an AM modulated signal as described in Example 7.10. We assume that $1 + mb(t) > 0$, that $b(t)$ is bandlimited to $[-B, B]$ and that $f_c > 2B$.

(a) Argue that the envelope of $|x(t)|$ is $(1 + mb(t))\sqrt{2}$ (a drawing will suffice).

(b) Prove that if we pass the signal $|x(t)|$ through an ideal lowpass filter of cutoff frequency f_0, we obtain $1 + mb(t)$ scaled by some factor. Specify a suitable interval for f_0.

Hint: Expand $|\cos(2\pi f_c t)|$ as a Fourier series. No need to find explicit values for the Fourier series coefficients.

(c) Argue that with a suitable choice of components, the output of the following circuit is essentially $b(t)$.

Hint: Draw, qualitatively, the voltage on top of R_1 and that on top of R_2.