SOLUTION 1.

(a) The state diagram and detour flow graph are shown here:
(b) Let \(a, b, c, d, e\) respectively represent the states \((1, 1), (-1, 1), (-1, -1), (1, -1)\) and \((1, 1)\). We have

\[
T_b = T_d I D + T_a I D^2 \\
T_c = T_c I D + T_b I D^2 \\
T_d = T_b D^2 + T_c D.
\]

Substituting \(T_c = T_b \frac{I D^2}{1 - I D}\) in the third equation above,

\[
T_a = T_b D^2 + T_b \frac{I D^3}{1 - I D}
= T_b \left( D^2 + \frac{I D^3}{1 - I D} \right)
= T_b \frac{D^2}{1 - I D}
= T_b \alpha,
\]

with \(\alpha = \frac{D^3}{1 - I D}\). The detour flow graph can thus be simplified as follows:

![Diagram](image)

In \(T_b = T_d I D + T_a I D^2\), we substitute for \(T_d\) to get

\[
T_b = T_a \frac{I D^2(1 - I D)}{1 - I D - I D^3}.
\]

It follows that

\[
T_d = T_b \frac{D^2}{1 - I D} = T_a \frac{I D^4}{1 - I D - I D^3},
\]

and that

\[
T(I, D) = T_c = T_a \frac{I D^7}{1 - I D - I D^3}.
\]

Taking the derivative yields

\[
\frac{\partial T(I, D)}{\partial I} = \frac{D^7(1 - I D - I D^3) - I D^7(-D - D^3)}{(1 - I D - I D^3)^2} = \frac{D^7}{(1 - I D - I D^3)^2}.
\]

Therefore, we find

\[
P_b \leq \left. \frac{\partial T(I, D)}{\partial I} \right|_{I = 1, D = z}
= \frac{z^7}{(1 - z - z^3)^2}.
\]
where \( z = e^{-\frac{E_s}{N_0}} \). Since there are three channel symbols per source symbol, we find that \( E_s = E_b/3 \).

Solution 2.

(a) An implementation of the encoder will be as follows:

(b) The state diagram is shown here:

We use the following terminology: the state label is \( x, y \), where \( x \) is the “state of the even sub-sequence”, i.e. contains \( b_{2n-2} \), and \( y \) is the “state of the odd sub-sequence”, i.e. contains \( b_{2n-1} \). On the arrows, we only mark the outputs; the input required to make a particular transition is simply the next state, therefore we omitted it. Transitions are labeled with the value of \( x_{3n}, x_{3n+1}, x_{3n+2} \).
(c) We use
$$P_b \leq \frac{1}{k_0} \left. \frac{\partial T(I,D)}{\partial I} \right|_{I=1,D=z},$$
where \( z = e^{-\frac{E_s}{N_0}} \) and \( k_0 \) is the number of inputs per section of the trellis. In this problem, \( k_0 = 2 \). Since there are three channel symbols per two source symbols, we find that \( E_s = \frac{2E_b}{3} \).

From the state diagram we can derive the generating functions of the detour flow graph:

$$T(I,D) = D^3 T_{-11} + D^2 T_{-1-1} + DT_{1-1}$$

$$T_{1-1} = IDT_{-11} + IT_{-1-1} + ID^3 T_{1-1} + ID^2 T_{11}$$

$$T_{-1-1} = I^2 DT_{-11} + I^2 D^2 T_{-1-1} + I^2 DT_{1-1} + I^2 D^2 T_{11}$$

$$T_{11} = IDT_{1-1} + ID^2 T_{1-1} + IDT_{1-1} + ID^2 T_{11}.$$ 

Solving the system gives
$$T(I,D) = T_{11} \frac{D^2 I (D^6 I + D^5 I^2 - D^3 - D^4 I - D)}{-D^8 I^3 - D^4 I^2 + D^3 I + 2D^2 I^2 + D^2 I + DI^3 + DI^2 + DI - 1},$$

on which we can apply the formula above.

**Solution 3.**

(a) Since the state is \((b_{j-1}, b_{j-2})\), we need two shift registers. From the finite state machine, we can derive a table that relates the state \((b_{j-1}, b_{j-2})\) and the current input \( b_j \) with the two outputs \((x_{2j}, x_{2j+1})\):

<table>
<thead>
<tr>
<th>( b_j )</th>
<th>( b_{j-1} )</th>
<th>( b_{j-2} )</th>
<th>( x_{2j} )</th>
<th>( x_{2j+1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>-1</td>
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<td>-1</td>
<td>-1</td>
<td>1</td>
</tr>
</tbody>
</table>

We can easily notice that the column of \( x_{2j} \) is the same as the column of \( b_{j-2} \). Therefore, \( x_{2j} = b_{j-2} \). On the other hand, we see that \( x_{2j+1} = b_{j-1} \) if \( b_j = 1 \) and \( x_{2j+1} = -b_{j-1} \) if \( b_j = -1 \). Therefore \( x_{2j+1} = b_j \cdot b_{j-1} \), which gives us the following encoder:
(b) The detour flow graph (with respect to the all-one sequence) is shown below:

We have

\[ T_b = T_a I D + T_d I D^2 \]
\[ T_c = T_b I + T_c I D \]
\[ T_d = T_c D^2 + T_b D \]
\[ T_e = T_d D \]

The solution of this system is \( T_e = T_a \frac{I D^3}{1 - ID - ID^3} \). Hence,

\[
P_b \leq \left. \frac{\partial T(I,D)}{\partial I} \right|_{I=1, D=z} = \left. \frac{D^3(1 - ID - ID^3) + ID^3(D + D^3)}{(1 - ID - ID^3)^2} \right|_{I=1, D=z}
\]

where \( z = e^{-\frac{E_b}{2N_0}} \).

**Solution 4.**

(a) The decoder is the same as in the example we have seen in Chapter 6 once the following isomorphic mapping is applied: \( \{1 \rightarrow 0, -1 \rightarrow 1\} \). Figure 6.4 shows the trellis of the encoder.

(b) Given the observation \( y = (y_1, \ldots, y_n) \), the ML codeword is given by \( \arg \max_{x \in C} p(y|x) \) where \( C \) represents the set of codewords (i.e., the set of all possible paths on the trellis). Alternately, the ML codeword is given by \( \arg \max_{x \in C} \sum_{i=1}^n \log p(y_i|x_i). \)
Hence, a branch metric for the BEC is

$$\log p(y_i|x_i) = \begin{cases} 
\log \epsilon & \text{if } y_i =?, \\
\log(1 - \epsilon) & \text{if } y_i = x_i, \\
-\infty & \text{if } y_i = 1 - x_i.
\end{cases}$$

(c) Given the observation $(0, ?, ?, 1, 0, 1)$, one can compute the branch metric in the trellis. Note that we do not need to further elaborate paths with a $-\infty$ metric. The decoding results $(0, 1, 0)$.

(d) We refer to the example shown in Chapter 6, where we have the same encoder, but a different channel. We have seen that

$$P_b \leq \frac{z^5}{(1 - 2z)^2}.$$  

To determine $z$ we use the Bhattacharyya bound, which in our case is

$$z = \sum_{y \in \{0, 1, ?\}} \sqrt{P(y|1)P(y|0)} = \epsilon.$$  

Thus we have the following bound:

$$P_b \leq \frac{\epsilon^5}{(1 - 2\epsilon)^2}.$$