ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 23 Principles of Digital Communications
Problem Set 10 May 6, 2015

PROBLEM 1. *(Nyquist Pulse)* A communication system uses signals of the form

$$\sum_{l \in \mathbb{Z}} s_l p(t - lT),$$

where s_l takes values in some symbol alphabet and $p(t)$ is a finite-energy pulse. The transmitted signal is first filtered by a channel of impulse response $h(t)$ and then corrupted by additive white Gaussian noise of power spectral density $\frac{N_0}{2}$. The receiver front-end is a filter of impulse response $q(t)$.

(a) Neglecting the noise, show that the front-end filter output has the form

$$y(t) = \sum_{l \in \mathbb{Z}} s_l g(t - lT),$$

where $g(t) = (p \star h \star q)(t)$ and $\star$ denotes convolution.

(b) The necessary and sufficient (time-domain) condition that $g(t)$ has to fulfill so that the samples of $y(t)$ satisfy $y(lT) = s_l$, $l \in \mathbb{Z}$, is

$$g(lT) = \mathbb{1}\{l = 0\}.$$

A function $g(t)$ that fulfills this condition is called a Nyquist pulse of parameter T. Prove the following theorem:

THEOREM. (Nyquist Criterion for Nyquist Pulses) *The $\mathcal{L}_2$ pulse $g(t)$ is a Nyquist pulse (with parameter T) if and only if its Fourier transform $g_{\mathcal{F}}(f)$ fulfills Nyquist's criterion (with parameter T), i.e.,*

$$\text{l. i. m.} \sum_{l \in \mathbb{Z}} g_{\mathcal{F}}(f - \frac{l}{T}) = T, \qquad t \in \mathbb{R}.$$

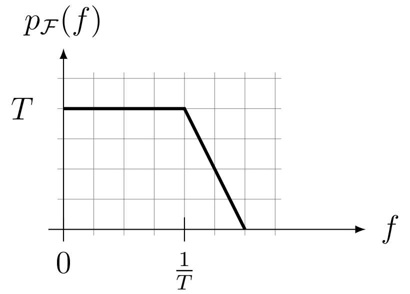
Note: Due to the periodicity of the left-hand side, equality is fulfilled if and only if it is fulfilled over an interval of length $1/T$.

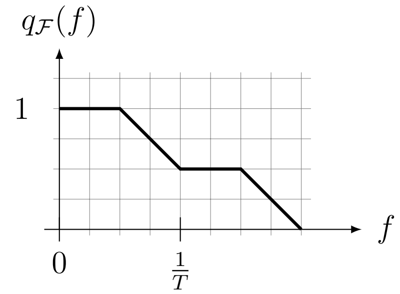
Hint: Set $g(t) = \int g_{\mathcal{F}}(f) e^{j2\pi ft} df$, insert on both sides $t = -lT$ and proceed as in the proof of Theorem 5.6.

(c) Prove Theorem 5.6 as a corollary to the above theorem.

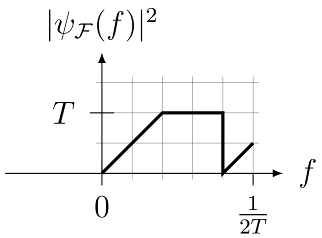
Hint: $\mathbb{1}\{l = 0\} = \int \psi(t - lT)\psi^*(t) dt$ if and only if the self-similarity function $R_\psi(\tau)$ is a Nyquist pulse with parameter T.

(d) Let $p(t)$ and $q(t)$ be real-valued with Fourier transform as shown below, where only positive frequencies are plotted (both functions being even). The channel frequency response is $h_{\mathcal{F}}(f) = 1$. Determine $y(kT),\ k \in \mathbb{Z}$.





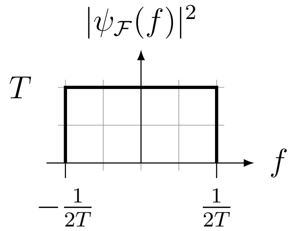
PROBLEM 2. *(Pulse Orthogonal to its T-Spaced Time Translates)* The following figure shows *part* of the plot of a function $|\psi_{\mathcal{F}}(f)|^2$, where $\psi_{\mathcal{F}}(f)$ is the Fourier transform of some pulse $\psi(t)$.



Complete the plot (for positive and negative frequencies) and label the ordinate, knowing that the following conditions are satisfied:

- For every pair of integers k, l, $\int \psi(t - kT)\psi(t - lT)dt = \mathbb{1}\{k = l\}$.
- $\psi(t)$ is real-valued.
- $\psi_{\mathcal{F}}(f) = 0$ for $|f| > \frac{1}{T}$.

PROBLEM 3. *(Peculiarity of Sinc)* Let $\psi(t)$ be the pulse with $|\psi_{\mathcal{F}}(f)|^2$ as depicted below.



(a) Given $|\psi_{\mathcal{F}}(f)|^2$, compute the time-domain pulse $\psi(t)$ and its self similarity function $R_{\psi}(t)$.

(b) Let $\{S_i\}_{i=-\infty}^{+\infty}$ be an i.i.d. sequence of uniformly distributed data symbols in $\{\pm 1\}$. Suppose the transmitter uses the pulse $\psi(t)$ to create the output waveform

$$W(t) = \sum_i S_i \psi(t - iT).$$

The signal $W(t)$ is then sent through an AWGN channel, i.e., the received signal is

$$R(t) = W(t) + N(t),$$

where $N(t)$ is the white Gaussian noise of power spectral density $\frac{N_0}{2}$.

The receiver filters $R(t)$ using a matched filter of impulse response $\psi^*(-t)$. Let $y(t)$ be the output of the matched filter. Show that if $y(t)$ is sampled at time $t = jT$ then the output samples are

$$Y_j = Y(jT) = S_j + Z_j$$

where $Z_j \sim \mathcal{N}(0, \frac{N_0}{2})$ are i.i.d. noise samples.

(c) Now suppose the receiver's clock has an offset of $\Delta \in (0, T)$ such that the matched filter's output is sampled at times $t = jT + \Delta$ instead of $t = jT$. Show that now the output samples are now

$$Y_j = l_0 S_j + \sum_{i \neq j} S_i l_{j-i} + Z_j,$$

where $Z_j \sim \mathcal{N}(0, \frac{N_0}{2})$ are i.i.d. noise samples. Compute the coefficients l_k.

(d) As you have seen in your book, the term $\sum_{i \neq j} S_i l_{j-i}$ is the ISI. Consider only the transmission of a finite block of data symbols $S_1, \dots, S_n$ (i.e. assume $S_i = 0$ for $i \notin \{1, 2, \dots, n\}$). Show that there exists a $\Delta \in (0, T)$ and particular realizations of this block such that the ISI contribution in Y_n can grow unboundedly with n.

Hint: The series $\sum_{k=1}^{\infty} \frac{1}{ak+b}$ *diverges.*

Problem 4. *(Power Spectral Density)* Consider the random process

$$X(t) = \sum_{i=-\infty}^{\infty} X_i \sqrt{\mathcal{E}_s} \psi(t - iT_s - T_0),$$

where T_s and $\mathcal{E}_s$ are fixed positive numbers, $\psi(t)$ is some unit-energy function, T_0 is a uniformly distributed random variable taking value in $[0, T_s)$, and $\{X_i\}_{i=-\infty}^{\infty}$ is the output of the convolutional encoder described by

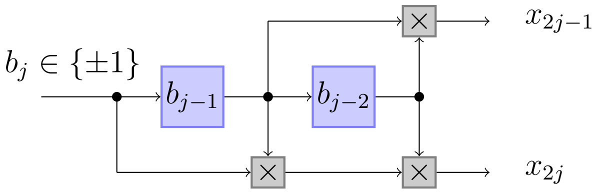
$$\begin{aligned} X_{2n} &= B_n B_{n-2} \\ X_{2n+1} &= B_n B_{n-1} B_{n-2} \end{aligned}$$

with i.i.d. input sequence $\{B_i\}_{i=-\infty}^{\infty}$ taking values in $\{\pm 1\}$.

(a) Express the power spectral density of $X(t)$ for a general $\psi(t)$.

(b) Plot the power spectral density of $X(t)$ assuming that $\psi(t)$ is a unit-norm rectangular pulse of width T_s.

PROBLEM 5. *(Viterbi Algorithm)* An output sequence $x_1, \dots, x_{10}$ from the convolutional encoder depicted in the figure below is transmitted over the discrete-time AWGN channel. The initial and final state of the encoder is $(1,1)$. Using the Viterbi algorithm, find the maximum likelihood information sequence $\hat{b}_1, \dots, \hat{b}_3, 1, 1$, knowing that $b_1, \dots, b_3$ are drawn independently and uniformly from $\{\pm 1\}$ and that the channel output $y_1, \dots, y_{10} = 1, 2, -1, 4, -2, 1, 1, -3, -1, -2$. (It is for convenience that we are choosing integers rather than real numbers.)



PROBLEM 6. *(Inter-Symbol Interference)* From the decoder's point of view, inter-symbol interference (ISI) can be modeled as follows

$$Y_i = X_i + Z_i$$
$$X_i = \sum_{j=0}^{L} B_{i-j} h_j, \qquad i = 1, 2, \dots \tag{1}$$

where B_i is the i-th information bit, $h_0, \dots, h_L$ are coefficients that describe the inter-symbol interference, and Z_i is zero-mean, Gaussian, of variance σ^2 and statistically independent of everything else. Relationship (1) can be described by a trellis, and the ML decision rule can be implemented by the Viterbi algorithm.

(a) Draw the trellis that describes all sequences of the form $X_1, \dots, X_6$ resulting from information sequences of the form $B_1, \dots, B_5, 0$, $B_i \in \{0, 1\}$, assuming

$$h_i = \begin{cases} 1, & i = 0 \\ -2, & i = 1 \\ 0, & \text{otherwise.} \end{cases}$$

To determine the initial state, you may assume that the preceding information sequence terminated with 0. Label the trellis edges with the input/output symbols.

(b) Specify a metric $f(x_1, \dots, x_6) = \sum_{i=1}^{6} f(x_i, y_i)$ whose minimization or maximization with respect to the valid $x_1, \dots, x_6$ leads to a maximum likelihood decision. Specify if your metric needs to be minimized or maximized.

(c) Assume $y_1, \dots, y_6 = 2, 0, -1, 1, 0, -1$. Find the maximum likelihood estimate of the information sequence $B_1, \dots, B_5$.