

PROBLEM 1.

(a) We have

$$\begin{aligned}
 W^-(y_1, y_2|u_1) &= \mathbb{P}_{Y_1, Y_2|X_1 \oplus X_2}(y_1, y_2|u_1) = \frac{\mathbb{P}_{Y_1, Y_2, X_1 \oplus X_2}(y_1, y_2, u_1)}{\mathbb{P}_{X_1 \oplus X_2}(u_1)} \\
 &\stackrel{(*)}{=} 2\mathbb{P}_{Y_1, Y_2, X_1 \oplus X_2}(y_1, y_2, u_1) \\
 &= 2 \sum_{u_2 \in \{0,1\}} \mathbb{P}_{Y_1, Y_2, X_1 \oplus X_2, X_2}(y_1, y_2, u_1, u_2) \\
 &\stackrel{(**)}{=} 2 \sum_{u_2 \in \{0,1\}} \mathbb{P}_{Y_1, Y_2, X_1, X_2}(y_1, y_2, u_1 \oplus u_2, u_2) \\
 &= 2 \sum_{u_2 \in \{0,1\}} \mathbb{P}_{Y_1, Y_2|X_1, X_2}(y_1, y_2|u_1 \oplus u_2, u_2) \mathbb{P}_{X_1, X_2}(u_1 \oplus u_2, u_2) \\
 &= 2 \sum_{u_2 \in \{0,1\}} W(y_1|u_1 \oplus u_2)W(y_2|u_2) \frac{1}{2^2} \\
 &= \frac{1}{2} \sum_{u_2 \in \{0,1\}} W(y_1|u_1 \oplus u_2)W(y_2|u_2),
 \end{aligned}$$

where (*) follows from the fact that if X_1, X_2 are independent and uniform then $X_1 \oplus X_2$ is also uniform. (**) follows from the fact that

$$(X_1 \oplus X_2 = u_1 \text{ and } X_2 = u_2) \Leftrightarrow (X_1 = u_1 \oplus u_2 \text{ and } X_2 = u_2).$$

(b) We have

$$\begin{aligned}
 W^+(y_1, y_2, u_1|u_2) &= \mathbb{P}_{Y_1, Y_2, X_1 \oplus X_2|X_2}(y_1, y_2, u_1|u_2) = \frac{\mathbb{P}_{Y_1, Y_2, X_1 \oplus X_2, X_2}(y_1, y_2, u_1, u_2)}{\mathbb{P}_{X_2}(u_2)} \\
 &= 2\mathbb{P}_{Y_1, Y_2, X_1 \oplus X_2, X_2}(y_1, y_2, u_1, u_2) \\
 &\stackrel{(*)}{=} 2\mathbb{P}_{Y_1, Y_2, X_1, X_2}(y_1, y_2, u_1 \oplus u_2, u_2) \\
 &= 2\mathbb{P}_{Y_1, Y_2|X_1, X_2}(y_1, y_2|u_1 \oplus u_2, u_2) \mathbb{P}_{X_1, X_2}(u_1 \oplus u_2, u_2) \\
 &= 2W(y_1|u_1 \oplus u_2)W(y_2|u_2) \frac{1}{2^2} \\
 &= \frac{1}{2}W(y_1|u_1 \oplus u_2)W(y_2|u_2),
 \end{aligned}$$

where (*) follows from the fact that

$$(X_1 \oplus X_2 = u_1 \text{ and } X_2 = u_2) \Leftrightarrow (X_1 = u_1 \oplus u_2 \text{ and } X_2 = u_2).$$

(c) We have

$$\begin{aligned}
Z(W^+) &= \sum_{\substack{y_1, y_2 \in \mathcal{Y}, \\ u_1 \in \{0,1\}}} \sqrt{W^+(y_1, y_2, u_1|0)W^+(y_1, y_2, u_1|1)} \\
&= \frac{1}{2} \sum_{\substack{y_1, y_2 \in \mathcal{Y}, \\ u_1 \in \{0,1\}}} \sqrt{W(y_1|u_1 \oplus 0)W(y_2|0)W(y_1|u_1 \oplus 1)W(y_2|1)} \\
&= \frac{1}{2} \left(\sum_{y_1, y_2 \in \mathcal{Y}} \sqrt{W(y_1|0 \oplus 0)W(y_2|0)W(y_1|0 \oplus 1)W(y_2|1)} \right) \\
&\quad + \frac{1}{2} \left(\sum_{y_1, y_2 \in \mathcal{Y}} \sqrt{W(y_1|1 \oplus 0)W(y_2|0)W(y_1|1 \oplus 1)W(y_2|1)} \right) \\
&= \frac{1}{2} \left(\sum_{y_1, y_2 \in \mathcal{Y}} \sqrt{W(y_1|0)W(y_2|0)W(y_1|1)W(y_2|1)} \right) \\
&\quad + \frac{1}{2} \left(\sum_{y_1, y_2 \in \mathcal{Y}} \sqrt{W(y_1|1)W(y_2|0)W(y_1|0)W(y_2|1)} \right) \\
&= \frac{1}{2} \left(\sum_{y_1 \in \mathcal{Y}} \sqrt{W(y_1|0)W(y_1|1)} \right) \left(\sum_{y_2 \in \mathcal{Y}} \sqrt{W(y_2|0)W(y_2|1)} \right) \\
&\quad + \frac{1}{2} \left(\sum_{y_1 \in \mathcal{Y}} \sqrt{W(y_1|0)W(y_1|1)} \right) \left(\sum_{y_2 \in \mathcal{Y}} \sqrt{W(y_2|0)W(y_2|1)} \right) \\
&= \frac{1}{2} Z(W) \cdot Z(W) + \frac{1}{2} Z(W) \cdot Z(W) = Z(W)^2.
\end{aligned}$$

(d) For every $y_1, y_2 \in \mathcal{Y}$, we have:

$$\begin{aligned}
W^-(y_1, y_2|0) &= \frac{1}{2} \sum_{u_2 \in \{0,1\}} W(y_1|0 \oplus u_2)W(y_2|u_2) = \frac{1}{2} \sum_{u_2 \in \{0,1\}} W(y_1|u_2)W(y_2|u_2) \\
&= \frac{1}{2} W(y_1|0)W(y_2|0) + \frac{1}{2} W(y_1|1)W(y_2|1) = \frac{1}{2} \alpha(y_1)\alpha(y_2) + \frac{1}{2} \beta(y_1)\beta(y_2) \\
&= \frac{1}{2} (\alpha(y_1)\alpha(y_2) + \beta(y_1)\beta(y_2)),
\end{aligned}$$

and

$$\begin{aligned}
W^-(y_1, y_2|1) &= \frac{1}{2} \sum_{u_2 \in \{0,1\}} W(y_1|1 \oplus u_2)W(y_2|u_2) \\
&= \frac{1}{2} W(y_1|1 \oplus 0)W(y_2|0) + \frac{1}{2} W(y_1|1 \oplus 1)W(y_2|1) \\
&= \frac{1}{2} W(y_1|1)W(y_2|0) + \frac{1}{2} W(y_1|0)W(y_2|1) = \frac{1}{2} \beta(y_1)\alpha(y_2) + \frac{1}{2} \alpha(y_1)\beta(y_2) \\
&= \frac{1}{2} (\alpha(y_1)\beta(y_2) + \beta(y_1)\alpha(y_2)).
\end{aligned}$$

We have

$$\begin{aligned} Z(W^-) &= \sum_{y_1, y_2 \in \mathcal{Y}} \sqrt{W^-(y_1, y_2|0)W^-(y_1, y_2|1)} \\ &= \frac{1}{2} \sum_{y_1, y_2 \in \mathcal{Y}} \sqrt{\left(\alpha(y_1)\alpha(y_2) + \beta(y_1)\beta(y_2)\right)\left(\alpha(y_1)\beta(y_2) + \beta(y_1)\alpha(y_2)\right)}. \end{aligned}$$

(e) For every $x, y \geq 0$, we have $x + y \leq x + y + 2\sqrt{xy} = (\sqrt{x} + \sqrt{y})^2$ which implies that $\sqrt{x + y} \leq \sqrt{x} + \sqrt{y}$. Therefore, for every $x, y, z, t \geq 0$ we have:

$$\sqrt{x + y + z + t} \leq \sqrt{x + y} + \sqrt{z + t} \leq \sqrt{x} + \sqrt{y} + \sqrt{z} + \sqrt{t}.$$

Therefore,

$$\begin{aligned} Z(W^-) &= \frac{1}{2} \sum_{y_1, y_2 \in \mathcal{Y}} \sqrt{\left(\alpha(y_1)\alpha(y_2) + \beta(y_1)\beta(y_2)\right)\left(\alpha(y_1)\beta(y_2) + \beta(y_1)\alpha(y_2)\right)} \\ &= \frac{1}{2} \sum_{y_1, y_2 \in \mathcal{Y}} \sqrt{\alpha(y_1)^2\gamma(y_2)^2 + \alpha(y_2)^2\gamma(y_1)^2 + \beta(y_2)^2\gamma(y_1)^2 + \beta(y_1)^2\gamma(y_2)^2} \\ &\stackrel{(*)}{\leq} \frac{1}{2} \sum_{y_1, y_2 \in \mathcal{Y}} \left(\sqrt{\alpha(y_1)^2\gamma(y_2)^2} + \sqrt{\alpha(y_2)^2\gamma(y_1)^2} + \sqrt{\beta(y_2)^2\gamma(y_1)^2} + \sqrt{\beta(y_1)^2\gamma(y_2)^2}\right) \\ &= \frac{1}{2} \left(\sum_{y_1, y_2 \in \mathcal{Y}} \alpha(y_1)\gamma(y_2)\right) + \frac{1}{2} \left(\sum_{y_1, y_2 \in \mathcal{Y}} \alpha(y_2)\gamma(y_1)\right) \\ &\quad + \frac{1}{2} \left(\sum_{y_1, y_2 \in \mathcal{Y}} \beta(y_2)\gamma(y_1)\right) + \frac{1}{2} \left(\sum_{y_1, y_2 \in \mathcal{Y}} \beta(y_1)\gamma(y_2)\right), \end{aligned}$$

where (*) follows from the inequality $\sqrt{x + y + z + t} \leq \sqrt{x} + \sqrt{y} + \sqrt{z} + \sqrt{t}$.

(f) Note that $\sum_{y \in \mathcal{Y}} \alpha(y) = \sum_{y \in \mathcal{Y}} \beta(y) = 1$ and $\sum_{y \in \mathcal{Y}} \gamma(y) = Z(W)$. Therefore,

$$\begin{aligned} Z(W^-) &\leq \frac{1}{2} \left(\sum_{y_1, y_2 \in \mathcal{Y}} \alpha(y_1)\gamma(y_2)\right) + \frac{1}{2} \left(\sum_{y_1, y_2 \in \mathcal{Y}} \alpha(y_2)\gamma(y_1)\right) \\ &\quad + \frac{1}{2} \left(\sum_{y_1, y_2 \in \mathcal{Y}} \beta(y_2)\gamma(y_1)\right) + \frac{1}{2} \left(\sum_{y_1, y_2 \in \mathcal{Y}} \beta(y_1)\gamma(y_2)\right) \\ &= \frac{1}{2} \left(\sum_{y_1 \in \mathcal{Y}} \alpha(y_1)\right) \left(\sum_{y_2 \in \mathcal{Y}} \gamma(y_2)\right) + \frac{1}{2} \left(\sum_{y_2 \in \mathcal{Y}} \alpha(y_2)\right) \left(\sum_{y_1 \in \mathcal{Y}} \gamma(y_1)\right) \\ &\quad + \frac{1}{2} \left(\sum_{y_2 \in \mathcal{Y}} \beta(y_2)\right) \left(\sum_{y_1 \in \mathcal{Y}} \gamma(y_1)\right) + \frac{1}{2} \left(\sum_{y_1 \in \mathcal{Y}} \beta(y_1)\right) \left(\sum_{y_2 \in \mathcal{Y}} \gamma(y_2)\right) \\ &= \frac{1}{2} 1 \cdot Z(W) + \frac{1}{2} 1 \cdot Z(W) + \frac{1}{2} 1 \cdot Z(W) + \frac{1}{2} 1 \cdot Z(W) = 2Z(W). \end{aligned}$$

PROBLEM 2.

(a) We have

$$\begin{aligned}
Q_{i+1} &= \sqrt{Z_{i+1}(1 - Z_{i+1})} = \begin{cases} \sqrt{Z_i^2(1 - Z_i^2)} & \text{w.p. } 1/2 \\ \sqrt{(2Z_i - Z_i^2)(1 - 2Z_i + Z_i^2)} & \text{w.p. } 1/2 \end{cases} \\
&= \begin{cases} \sqrt{Z_i^2(1 - Z_i)(1 + Z_i)} & \text{w.p. } 1/2 \\ \sqrt{(2 - Z_i)Z_i(1 - Z_i)^2} & \text{w.p. } 1/2 \end{cases} \\
&= \begin{cases} \sqrt{Z_i(1 - Z_i)}\sqrt{Z_i(1 + Z_i)} & \text{w.p. } 1/2 \\ \sqrt{Z_i(1 - Z_i)}\sqrt{(2 - Z_i)(1 - Z_i)} & \text{w.p. } 1/2 \end{cases} \\
&= \sqrt{Z_i(1 - Z_i)} \begin{cases} \sqrt{Z_i(1 + Z_i)} & \text{w.p. } 1/2 \\ \sqrt{(2 - Z_i)(1 - Z_i)} & \text{w.p. } 1/2 \end{cases} \\
&= Q_i \begin{cases} f_1(Z_i) & \text{w.p. } 1/2 \\ f_2(Z_i) & \text{w.p. } 1/2 \end{cases},
\end{aligned}$$

where $f_1(z) = \sqrt{z(z+1)}$ and $f_2(z) = \sqrt{(2-z)(1-z)}$.

(b) We have

$$f_1'(z) = \frac{2z+1}{2\sqrt{z(z+1)}}$$

so

$$\begin{aligned}
f_1''(z) &= \frac{4\sqrt{z(z+1)} - (2z+1)\frac{2(2z+1)}{2\sqrt{z(z+1)}}}{\left(2\sqrt{z(z+1)}\right)^2} \\
&= \frac{4z(z+1) - (2z+1)^2}{4(z(z+1))^{\frac{3}{2}}} = \frac{-1}{4(z(z+1))^{\frac{3}{2}}} \leq 0.
\end{aligned}$$

Therefore, f_1 is concave. By noticing that $f_2(z) = f_1(1-z)$, we obtain:

$$\begin{aligned}
f_1(z) + f_2(z) &= f_1(z) + f_1(1-z) = 2 \left(\frac{1}{2}f_1(z) + \frac{1}{2}f_1(1-z) \right) \\
&\stackrel{(*)}{\leq} 2f_1 \left(\frac{1}{2}z + \frac{1}{2}(1-z) \right) = 2f_1 \left(\frac{1}{2} \right) = 2\sqrt{\frac{1}{2} \left(\frac{1}{2} + 1 \right)} \\
&= 2\sqrt{\frac{1}{2} \cdot \frac{3}{2}} = 2\frac{\sqrt{3}}{2} = \sqrt{3},
\end{aligned}$$

where (*) follows from the concavity of f_1 . We have

$$\mathbb{E}[Q_{i+1} \mid Z_0, \dots, Z_i] = \frac{1}{2}f_1(Z_i)Q_i + \frac{1}{2}f_2(Z_i)Q_i = \frac{1}{2}(f_1(Z_i) + f_2(Z_i))Q_i \leq \rho Q_i,$$

where $\rho = \frac{\sqrt{3}}{2} < 1$.

(c) We will show the claim by induction on $i \geq 0$. For $i = 0$, we have $Z_0 = z_0$ with probability 1. Therefore, $\mathbb{E}Q_0 = \sqrt{z_0(1-z_0)}$.

It is easy to see that the function $[0, 1] \rightarrow \mathbb{R}$ defined by $z \rightarrow \sqrt{z(1-z)}$ achieves its maximum at $z = \frac{1}{2}$, and so $\mathbb{E}Q_0 = \sqrt{z_0(1-z_0)} \leq \sqrt{\frac{1}{2} \left(1 - \frac{1}{2}\right)} = \frac{1}{2}$. Therefore, the claim is true for $i = 0$.

Now suppose that the claim is true for $i \geq 0$, i.e., $\mathbb{E}Q_i \leq \frac{1}{2}\rho^i$. We have

$$\mathbb{E}Q_{i+1} = \mathbb{E}[\mathbb{E}[Q_{i+1} \mid Z_0, \dots, Z_i]] \stackrel{(*)}{\leq} \mathbb{E}[\rho Q_i] = \rho \mathbb{E}[Q_i] \stackrel{(**)}{\leq} \rho \cdot \frac{1}{2}\rho^i = \frac{1}{2}\rho^{i+1},$$

where $(*)$ follows from Part (b) and $(**)$ follows from the induction hypothesis. We conclude that $\mathbb{E}Q_i \leq \frac{1}{2}\rho^i$ for every $i \geq 0$.

(d) By noticing that $\delta < z < 1 - \delta$ if and only if $z(1-z) > \delta(1-\delta)$, we get:

$$\begin{aligned} \mathbb{P}[Z_i \in (\delta, 1 - \delta)] &= \mathbb{P}[Z_i(1 - Z_i) > \delta(1 - \delta)] = \mathbb{P}[\sqrt{Z_i(1 - Z_i)} > \sqrt{\delta(1 - \delta)}] \\ &= \mathbb{P}[Q_i > \sqrt{\delta(1 - \delta)}] \stackrel{(*)}{\leq} \frac{\mathbb{E}Q_i}{\sqrt{\delta(1 - \delta)}} \stackrel{(**)}{\leq} \frac{\rho^i}{2\sqrt{\delta(1 - \delta)}}, \end{aligned}$$

where $(*)$ follows from the Markov inequality and $(**)$ follows from Part (c). Now since $\rho < 1$, we have $\frac{\rho^i}{2\sqrt{\delta(1 - \delta)}} \rightarrow 0$ as $i \rightarrow \infty$. We conclude that

$$\mathbb{P}[Z_i \in (\delta, 1 - \delta)] \rightarrow 0 \text{ as } i \text{ gets large.}$$

PROBLEM 3.

(a)

$$\begin{aligned} I(X; Y) &= I(X_k, K; Y_k, K) = I(K; Y_k, K) + I(X_k; Y_k, K \mid K) = H(K) + I(X_k; Y_k \mid K) \\ &= h_2(\alpha) + \mathbb{P}_K[1] \cdot I(X_k; Y_k \mid K = 1) + \mathbb{P}_K[2] \cdot I(X_k; Y_k \mid K = 2) \\ &= h_2(\alpha) + \alpha \cdot I(X_1; Y_1) + (1 - \alpha) \cdot I(X_2; Y_2). \end{aligned}$$

(b) The distribution of X is determined by α and by the distributions of X_1 and X_2 . It is clear from the expression in (a) that for any given α , $I(X; Y)$ is maximized when $I(X_1; Y_1)$ and $I(X_2; Y_2)$ are maximized, i.e., when the distribution of X_1 (resp. X_2) achieves the capacity of P_1 (resp. P_2). We conclude that the value of α in the capacity achieving distribution is the one that maximizes the function $f(\alpha) = h_2(\alpha) + \alpha C_1 + (1 - \alpha) C_2$. The derivative of f is:

$$f'(\alpha) = -\log_2(\alpha) - \frac{1}{\ln 2} + \log_2(1 - \alpha) + \frac{1}{\ln 2} + C_1 - C_2 = C_1 - C_2 + \log_2 \frac{1 - \alpha}{\alpha}.$$

We have $f'(\alpha) = 0$ (resp. $f'(\alpha) > 0$, $f'(\alpha) < 0$) if $\alpha = \alpha^*$ (resp. $\alpha < \alpha^*$, $\alpha > \alpha^*$), where $\alpha^* = \frac{2^{C_1}}{2^{C_1} + 2^{C_2}}$. This means that $f(\alpha)$ is maximized at $\alpha = \alpha^*$. Therefore,

the capacity achieving distribution is such that $\alpha = \frac{2^{C_1}}{2^{C_1} + 2^{C_2}}$ and X_1 (resp. X_2) achieves the capacity of the channel P_1 (resp. P_2).

(c) From (b), we have:

$$\begin{aligned} C &= -\frac{2^{C_1}}{2^{C_1} + 2^{C_2}} \log_2 \frac{2^{C_1}}{2^{C_1} + 2^{C_2}} - \frac{2^{C_2}}{2^{C_1} + 2^{C_2}} \log_2 \frac{2^{C_2}}{2^{C_1} + 2^{C_2}} + \frac{2^{C_1} C_1}{2^{C_1} + 2^{C_2}} + \frac{2^{C_2} C_2}{2^{C_1} + 2^{C_2}} \\ &= -\frac{2^{C_1}}{2^{C_1} + 2^{C_2}} C_1 + \frac{2^{C_1}}{2^{C_1} + 2^{C_2}} \log_2(2^{C_1} + 2^{C_2}) - \frac{2^{C_2}}{2^{C_1} + 2^{C_2}} C_2 \\ &\quad + \frac{2^{C_2}}{2^{C_1} + 2^{C_2}} \log_2(2^{C_1} + 2^{C_2}) + \frac{2^{C_1} C_1}{2^{C_1} + 2^{C_2}} + \frac{2^{C_2} C_2}{2^{C_1} + 2^{C_2}} \\ &= \log_2(2^{C_1} + 2^{C_2}). \end{aligned}$$

Therefore, $2^C = 2^{C_1} + 2^{C_2}$.