ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 25 Information Theory and Coding
Solutions to homework 10 December 09, 2014

PROBLEM 1.

(a) We have

$$\begin{aligned}\mathbb{P}[U(1) \neq U^n | U^n = u^n] &= \mathbb{P}[U(1) \neq u^n | U^n = u^n] \overset{(*)}{=} \mathbb{P}[U(1) \neq u^n] \\ &= 1 - \mathbb{P}[U(1) = u^n] = 1 - \prod_{i=1}^{n} \mathbb{P}[U(1)_i = u_i] = 1 - \prod_{i=1}^{n} p_U(u_i),\end{aligned}$$

where $(*)$ follows from the independence of $U(1)$ and U^n.

(b) An encoding failure happens if and only if $U(m) \neq U^n$ for every $1 \leq m \leq M$. Therefore,

$$\begin{aligned}\mathbb{P}[\text{“failure”} | U^n = u^n] &= \mathbb{P}[U(m) \neq U^n,\ \forall 1 \leq m \leq M | U^n = u^n] \\ &= \mathbb{P}[U(m) \neq u^n,\ \forall 1 \leq m \leq M | U^n = u^n] \\ &= \mathbb{P}[U(m) \neq u^n,\ \forall 1 \leq m \leq M] \\ &= \prod_{m=1}^{M} \left(1 - \prod_{i=1}^{n} p_U(u_i)\right) = \left(1 - \prod_{i=1}^{n} p_U(u_i)\right)^M.\end{aligned}$$

(c) Note that if $u^n \in \mathcal{T}_\epsilon^n(p_U)$, then $\prod_{i=1}^{n} p_U(u_i) \geq 2^{-nH(U)(1+\epsilon)}$, which implies that

$$\begin{aligned}\mathbb{P}[\text{“failure”} | U^n = u^n] &= \left(1 - \prod_{i=1}^{n} p_U(u_i)\right)^M \leq (1 - 2^{-nH(U)(1+\epsilon)})^M \\ &\overset{(*)}{\leq} \left(e^{-2^{-nH(U)(1+\epsilon)}}\right)^M = e^{-M2^{-nH(U)(1+\epsilon)}} = e^{-2^{nR-nH(U)(1+\epsilon)}},\end{aligned}$$

where $(*)$ follows from the inequality $1 - x \leq e^{-x}$. Therefore, we have

$$\begin{aligned}\mathbb{P}[\text{“failure”} | U^n \in \mathcal{T}_\epsilon^n(p_U)] &= \frac{\mathbb{P}[\text{“failure”}, U^n \in \mathcal{T}_\epsilon^n(p_U)]}{\mathbb{P}[U^n \in \mathcal{T}_\epsilon^n(p_U)]} \\ &= \frac{\sum_{u^n \in \mathcal{T}_\epsilon^n(p_U)} \mathbb{P}[\text{“failure”}, U^n = u^n]}{\mathbb{P}[U^n \in \mathcal{T}_\epsilon^n(p_U)]} \\ &= \frac{\sum_{u^n \in \mathcal{T}_\epsilon^n(p_U)} \mathbb{P}[\text{“failure”} | U^n = u^n]\mathbb{P}[U^n = u^n]}{\mathbb{P}[U^n \in \mathcal{T}_\epsilon^n(p_U)]} \\ &\leq \frac{\sum_{u^n \in \mathcal{T}_\epsilon^n(p_U)} e^{-2^{nR-nH(U)(1+\epsilon)}}\mathbb{P}[U^n = u^n]}{\mathbb{P}[U^n \in \mathcal{T}_\epsilon^n(p_U)]} \\ &= e^{-2^{nR-nH(U)(1+\epsilon)}} \frac{\sum_{u^n \in \mathcal{T}_\epsilon^n(p_U)} \mathbb{P}[U^n = u^n]}{\mathbb{P}[U^n \in \mathcal{T}_\epsilon^n(p_U)]} \\ &= e^{-2^{nR-nH(U)(1+\epsilon)}} \frac{\mathbb{P}[U^n \in \mathcal{T}_\epsilon^n(p_U)]}{\mathbb{P}[U^n \in \mathcal{T}_\epsilon^n(p_U)]} = e^{-2^{nR-nH(U)(1+\epsilon)}}.\end{aligned}$$

(d) Assume that $R > H(U)$, then there exists $\epsilon > 0$ such that $R > H(U) + \epsilon$. We have

$$\begin{aligned}
\mathbb{P}[\text{“failure”}] &= \mathbb{P}[\text{“failure”}, U^n \in \mathcal{T}_\epsilon^n(p_U)] + \mathbb{P}[\text{“failure”}, U^n \in \mathcal{T}_\epsilon^n(p_U)^c] \\
&= \mathbb{P}[\text{“failure”}|U^n \in \mathcal{T}_\epsilon^n(p_U)]\mathbb{P}[U^n \in \mathcal{T}_\epsilon^n(p_U)] + \mathbb{P}[\text{“failure”}, U^n \in \mathcal{T}_\epsilon^n(p_U)^c] \\
&\leq \mathbb{P}[\text{“failure”}|U^n \in \mathcal{T}_\epsilon^n(p_U)] + \mathbb{P}[U^n \in \mathcal{T}_\epsilon^n(p_U)^c] \\
&\leq e^{-2^{nR-nH(U)(1+\epsilon)}} + \mathbb{P}_{U^n}(\mathcal{T}_\epsilon^n(p_U)^c).
\end{aligned}$$

On the other hand, $\mathbb{P}_{U^n}(\mathcal{T}_\epsilon^n(p_U)^c) \to 0$ as $n \to \infty$, and $e^{-2^{nR-nH(U)(1+\epsilon)}} \to 0$ as $n \to \infty$ since $R > H(U) + \epsilon$. Therefore, if $R > H(U)$ then $\mathbb{P}[\text{“failure”}] \to 0$ as $n \to \infty$.

Problem 2.

(a) For every $0 \leq p \leq 1$, define $\overline{p} := 1 - p$. We have:

$$h_2(\overline{p}) = -\overline{p}\log\overline{p} - p\log p = -p\log p - \overline{p}\log\overline{p} = h_2(p). \tag{1}$$

On the other hand, it is easy to check that for every $0 \leq p', p'' \leq 1$, we have:

$$\overline{p'} * p'' = p' * \overline{p''} = \overline{p' * p''} \quad \text{and} \quad \overline{p'} * \overline{p''} = p' * p''.$$

Now (1) implies that

$$h_2(\overline{p'} * p'') = h_2(p' * \overline{p''}) = h_2(\overline{p'} * \overline{p''}) = h_2(p' * p''). \tag{2}$$

Let $p' = \mathbb{P}[X_1 = 1]$ and $p'' = \mathbb{P}[X_2 = 1]$. We have the following:

- $\mathbb{P}[X_1 \oplus X_2 = 1] = \mathbb{P}[X_1 = 1]\mathbb{P}[X_2 = 0] + \mathbb{P}[X_1 = 0]\mathbb{P}[X_2 = 1] = p'\overline{p''} + \overline{p'}p'' = p' * p''$. Therefore, $H(X_1 * X_2) = h_2(p' * p'')$.
- Since $H(X_1) = h_2(p_1)$, then we have either $p' = p_1$ or $p' = 1 - p_1$. I.e., we have $p_1 = p'$ or $p_1 = 1 - p' = \overline{p'}$.
- Since $H(X_2) = h_2(p_2)$, then we have either $p'' = p_2$ or $p'' = 1 - p_2$. I.e., we have $p_2 = p''$ or $p_2 = 1 - p'' = \overline{p''}$.

Now (2) implies that $H(X_1 \oplus X_2) = h_2(p' * p'') = h_2(p_1 * p_2)$.

(b) We have $H(X_1|Y) = \sum_{y\in\mathcal{Y}} H(X_1|Y = y)\mathbb{P}_Y(y) = \sum_{y\in\mathcal{Y}} h_2(p_1(y))q(y)$.

Now for every $y \in \mathcal{Y}$, X_1 and X_2 are independent conditioned on $Y = y$. Moreover, $H(X_1|Y = y) = h_2(p_1(y))$ and $H(X_2|Y = y) = H(X_2) = h_2(p_2)$ since X_2 and Y are independent. Therefore, Part (a) implies that $H(X_1 \oplus X_2|Y = y) = h_2(p_1(y) * p_2)$.

We conclude that

$$\begin{aligned}
H(X_1 \oplus X_2|Y) &= \sum_{y\in\mathcal{Y}} H(X_1 \oplus X_2|Y = y)\mathbb{P}_Y(y) \\
&= \sum_{y\in\mathcal{Y}} h_2(p_1(y) * p_2)q(y) = \sum_{y\in\mathcal{Y}} h_2(p_2 * p_1(y))q(y).
\end{aligned}$$

(c) Note that $p_2 * p = p(1 - p_2) + p_2(1 - p) = \beta p + p_2$, where $\beta = 1 - 2p_2 \geq 0$. Let $g(p) = \frac{\frac{\partial}{\partial p}h_2(p_2 * p)}{\frac{\partial}{\partial p}h_2(p)} = \frac{\frac{\partial}{\partial p}h_2(\beta p + p_2)}{\frac{\partial}{\partial p}h_2(p)} = \frac{\beta h_2'(\beta p + p_2)}{h_2'(p)}$. We have

$$\begin{aligned}
g'(p) &= \frac{\beta^2 h_2''(\beta p + p_2)h_2'(p) - \beta h_2''(p)h_2'(\beta p + p_2)}{h_2'(p)^2} \\
&= \frac{\beta h_2''(\beta p + p_2)h_2''(p)}{h_2'(p)^2}\Big[\beta\frac{h_2'(p)}{h_2''(p)} - \frac{h_2'(\beta p + p_2)}{h_2''(\beta p + p_2)}\Big].
\end{aligned}$$

Note that $h_2'(p) = \log \frac{1-p}{p}$ and $h_2''(p) = \frac{-1}{p(1-p)\ln 2}$, which implies that $h_2''(\beta p + p_2) \le 0$ and $h_2''(p) \le 0$. Therefore, $\frac{\beta h_2''(\beta p+p_2) h_2''(p)}{h_2'(p)^2} \ge 0$ and so it is sufficient to show that we have $\beta \frac{h_2'(p)}{h_2''(p)} - \frac{h_2'(\beta p+p_2)}{h_2''(\beta p+p_2)} \ge 0$. Now define $\alpha = 1 - 2p$. It is easy to check the following:

- $p = \frac{1}{2}(1-\alpha)$.
- $1 - p = \frac{1}{2}(1+\alpha)$.
- $\beta p + p_2 = \frac{1}{2}(1 - \alpha\beta)$.
- $1 - (\beta p + p_2) = \frac{1}{2}(1 + \alpha\beta)$.

Therefore, we have

$$\beta \frac{h_2'(p)}{h_2''(p)} = -\beta(\ln 2) p(1-p) \log \frac{1-p}{p} = -\frac{\beta \ln 2}{4}(1-\alpha^2) \log \frac{1+\alpha}{1-\alpha},$$

and

$$\frac{h_2'(\beta p + p_2)}{h_2''(\beta p + p_2)} = -(\ln 2)(\beta p + p_2)(1 - \beta p - p_2) \log \frac{1 - \beta p - p_2}{\beta p + p_2} = -\frac{\ln 2}{4}(1 - (\alpha\beta)^2) \log \frac{1 + \alpha\beta}{1 - \alpha\beta}.$$

Using the formula $\log(1+x) = \sum_{k \ge 1} (-1)^{k-1} \frac{x^k}{k}$, we get

$$\log \frac{1+x}{1-x} = \log(1+x) - \log(1-x) = \left(\sum_{k \ge 1} (-1)^{k-1} \frac{x^k}{k} \right) - \left(\sum_{k \ge 1} (-1)^{k-1} \frac{(-x)^k}{k} \right)$$
$$= \sum_{k \ge 1} \left((-1)^{k-1} + 1 \right) \frac{x^k}{k} = 2 \sum_{\substack{k \ge 1 \\ k \text{ is odd}}} \frac{x^k}{k}.$$

Therefore,

$$-(1-x^2) \log \frac{1+x}{1-x} = -2 \sum_{\substack{k \ge 1 \\ k \text{ is odd}}} \frac{x^k}{k} + 2 \sum_{\substack{k \ge 1 \\ k \text{ is odd}}} \frac{x^{k+2}}{k} = -2x - 2 \sum_{\substack{k \ge 3 \\ k \text{ is odd}}} \frac{x^k}{k} + 2 \sum_{\substack{k \ge 3 \\ k \text{ is odd}}} \frac{x^k}{k-2}$$
$$= -2x + 2 \sum_{\substack{k \ge 3 \\ k \text{ is odd}}} \left(\frac{1}{k-2} - \frac{1}{k} \right) x^k.$$

Hence,

$$\beta \frac{h_2'(p)}{h_2''(p)} = -\frac{\beta \ln 2}{4}(1-\alpha^2) \log \frac{1+\alpha}{1-\alpha} = \frac{\beta \ln 2}{4} \left[-2\alpha + 2 \sum_{\substack{k \ge 3 \\ k \text{ is odd}}} \left(\frac{1}{k-2} - \frac{1}{k} \right) \alpha^k \right]$$
$$= -\frac{\alpha\beta \ln 2}{2} + \frac{\ln 2}{2} \sum_{\substack{k \ge 3 \\ k \text{ is odd}}} \left(\frac{1}{k-2} - \frac{1}{k} \right) \beta\alpha^k,$$

and

$$\frac{h_2'(\beta p+p_2)}{h_2''(\beta p+p_2)} = -\frac{\ln 2}{4}(1-(\alpha\beta)^2)\log\frac{1+\alpha\beta}{1-\alpha\beta} = \frac{\ln 2}{4}\left[-2\alpha\beta + 2\sum_{\substack{k\geq 3\\ k \text{ is odd}}}\left(\frac{1}{k-2}-\frac{1}{k}\right)(\alpha\beta)^k\right]$$

$$= -\frac{\alpha\beta\ln 2}{2} + \frac{\ln 2}{2}\sum_{\substack{k\geq 3\\ k \text{ is odd}}}\left(\frac{1}{k-2}-\frac{1}{k}\right)\beta^k\alpha^k.$$

We conclude that

$$\beta\frac{h_2'(p)}{h_2''(p)} - \frac{h_2'(\beta p+p_2)}{h_2''(\beta p+p_2)} = \frac{\ln 2}{2}\sum_{\substack{k\geq 3\\ k \text{ is odd}}}\left(\frac{1}{k-2}-\frac{1}{k}\right)(\beta-\beta^k)\alpha^k \overset{(*)}{\geq} 0,$$

where $(*)$ follows from the fact that $\beta = 1-2p_2 \leq 1$ which implies that $\beta^k \leq \beta$. Therefore, $g'(p) \geq 0$ and so $g(p)$ is increasing. We conclude that the function f is convex.

(d) We have

$$\begin{aligned} H(X_1\oplus X_2|Y) &= \sum_{y\in\mathcal{Y}} h_2(p_2 * p_1(y))q(y) = \sum_{y\in\mathcal{Y}} h_2\big(p_2 * h_2^{-1}\big(H(X_1|Y=y)\big)\big)q(y) \\ &= \sum_{y\in\mathcal{Y}} f\big(H(X_1|Y=y)\big)q(y) \overset{(*)}{\geq} f\Big(\sum_{y\in\mathcal{Y}} H(X_1|Y=y)q(y)\Big) \\ &= f(H(X_1|Y)) = h_2\big(p_2 * h_2^{-1}\big(H(X_1|Y)\big)\big) = h_2(p_2 * p_1) = h_2(p_1 * p_2), \end{aligned}$$

where $(*)$ follows from the convexity of the function f.

(e) For every $y_1 \in \mathcal{Y}_1$, let $0 \leq p_1(y_1) \leq \frac{1}{2}$ be such that $H(X_1|Y_1 = y_1) = h_2(p_1(y_1))$ and let $q_1(y_1) = \mathbb{P}_{Y_1}(y_1)$. Similarly, for every $y_2 \in \mathcal{Y}_2$, let $0 \leq p_2(y_2) \leq \frac{1}{2}$ be such that $H(X_2|Y_2 = y_2) = h_2(p_2(y_2))$ and let $q_2(y_2) = \mathbb{P}_{Y_2}(y_2)$. For every $y_1 \in \mathcal{Y}_1$, define the mapping $f_{y_1} : [0,1] \to \mathbb{R}$ as $f_{y_1}(h) = h_2(p_1(y) * h_2^{-1}(h))$. Part (c) implies that f_{y_1} is convex for every $y_1 \in \mathcal{Y}_1$. We have

$$\begin{aligned} H(X_1\oplus X_2|Y_1,Y_2) &= \sum_{y_1\in\mathcal{Y}_1}\sum_{y_2\in\mathcal{Y}_2} h_2(p_1(y_1) * p_2(y_2))\mathbb{P}_{Y_1,Y_2}(y_1,y_2) \\ &= \sum_{y_1\in\mathcal{Y}_1}\sum_{y_2\in\mathcal{Y}_2} h_2(p_1(y_1) * p_2(y_2))q_1(y_1)q_2(y_2) \\ &= \sum_{y_1\in\mathcal{Y}_1} q_1(y_1)\sum_{y_2\in\mathcal{Y}_2} h_2\big(p_1(y_1) * h_2^{-1}\big(H(X_2|Y_2=y_2)\big)\big)q_2(y_2) \\ &= \sum_{y_1\in\mathcal{Y}_1} q_1(y_1)\sum_{y_2\in\mathcal{Y}_2} f_{y_1}\big(H(X_2|Y_2=y_2)\big)q_2(y_2) \\ &\overset{(*)}{\geq} \sum_{y_1\in\mathcal{Y}_1} q_1(y_1)f_{y_1}\Big(\sum_{y_2\in\mathcal{Y}_2} H(X_2|Y_2=y_2)q_2(y_2)\Big) \\ &= \sum_{y_1\in\mathcal{Y}_1} q_1(y_1)f_{y_1}(H(X_2|Y_2)) = \sum_{y_1\in\mathcal{Y}_1} q_1(y_1)h_2\big(p_1(y_1) * h_2^{-1}\big(H(X_2|Y_2)\big)\big) \\ &= \sum_{y_1\in\mathcal{Y}_1} q_1(y_1)h_2(p_1(y_1) * p_2) = \sum_{y_1\in\mathcal{Y}_1} h_2\big(p_2 * h_2^{-1}\big(H(X_1|Y_1=y_1)\big)\big)q_1(y_1) \\ &= \sum_{y_1\in\mathcal{Y}_1} f\big(H(X_1|Y_1=y_1)\big)q_1(y_1) \overset{(**)}{\geq} f\Big(\sum_{y_1\in\mathcal{Y}_1} H(X_1|Y_1=y_1)q(y_1)\Big) \\ &= f(H(X_1|Y_1)) = h_2\big(p_2 * h_2^{-1}\big(H(X_1|Y_1)\big)\big) = h_2(p_2 * p_1) = h_2(p_1 * p_2), \end{aligned}$$

where $(*)$ follows from the convexity of the functions $\{f_{y_1}:\ y_1\in\mathcal{Y}_1\}$ and $(**)$ follows from the convexity of f.

PROBLEM 3.

(a) Since $u^n\in\mathcal{T}_\delta^n(p_U)$, we have $n\mathbb{P}_U(a)(1-\delta)\leq n_a(u^n)\leq n\mathbb{P}_U(a)(1+\delta)$. Therefore, we have:

$$\begin{aligned} n_{a,b}(u^n,v^n) &\leq n_a(u^n)\mathbb{P}_{V|U}(b|a)(1+\delta)\leq n\mathbb{P}_U(a)(1+\delta)\mathbb{P}_{V|U}(b|a)(1+\delta)\\ &= n\mathbb{P}_{U,V}(a,b)(1+2\delta+\delta^2)\leq n\mathbb{P}_{U,V}(a,b)(1+3\delta), \end{aligned}$$

and

$$\begin{aligned} n_{a,b}(u^n,v^n) &\geq n_a(u^n)\mathbb{P}_{V|U}(b|a)(1-\delta)\geq n\mathbb{P}_U(a)(1-\delta)\mathbb{P}_{V|U}(b|a)(1-\delta)\\ &= n\mathbb{P}_{U,V}(a,b)(1-2\delta+\delta^2)\geq n\mathbb{P}_{U,V}(a,b)(1-3\delta). \end{aligned}$$

Therefore, $(u^n,v^n)\in\mathcal{T}_{3\delta}^n(p_{U,V})$.

(b) For every $a\in\mathcal{U}$, let v_a^n be the subsequence of v^n corresponding to the indices $1\leq i\leq n$ where $u_i=a$. Note that the size of the sequence v_a^n is $n_a(u^n)$ (i.e., $v_a^n\in\mathcal{V}^{n_a(u^n)}$). Moreover, for every $b\in\mathcal{V}$ we have $n_b(v_a^n)=n_{a,b}(u^n,v^n)$. Therefore, the condition

$$n_a(u^n)\mathbb{P}_{V|U}(b|a)(1-\delta)\leq n_{a,b}(u^n,v^n)\leq n_a(u^n)\mathbb{P}_{V|U}(b|a)(1+\delta)\quad\forall a\in\mathcal{U},\ \forall b\in\mathcal{V}\quad(3)$$

is equivalent to the the condition "$v_a^n\in\mathcal{T}_\delta^{n_a(u^n)}(p_{V|U=a})$ for every $a\in\mathcal{V}$". Now since $|\mathcal{T}_\delta^{n_a(u^n)}(p_{V|U=a})|\geq(1-\delta)2^{n_a(u^n)H(V|U=a)(1-\delta)}$ for every $a\in\mathcal{U}$, and since the correspondence $v^n\leftrightarrow(v_a^n)_{a\in\mathcal{U}}$ is a one-to-one correspondence, we conclude that there are at least $\prod_{a\in\mathcal{U}}\left[(1-\delta)2^{n_a(u^n)H(V|U=a)(1-\delta)}\right]$ sequences $v^n\in\mathcal{V}^n$ satisfying (3).

(c) Part (b) shows that there are at least $\prod_{a\in\mathcal{U}}\left[(1-\delta)2^{n_a(u^n)H(V|U=a)(1-\delta)}\right]$ sequences $v^n\in\mathcal{V}^n$ satisfying (3). We have

$$\begin{aligned} \prod_{a\in\mathcal{U}}\left[(1-\delta)2^{n_a(u^n)H(V|U=a)(1-\delta)}\right] &= (1-\delta)^{|\mathcal{U}|}\prod_{a\in\mathcal{U}}2^{n_a(u^n)H(V|U=a)(1-\delta)}\\ &= (1-\delta)^{|\mathcal{U}|}2^{\sum_{a\in\mathcal{U}}n_a(u^n)H(V|U=a)(1-\delta)}\\ &\geq (1-\delta)^{|\mathcal{U}|}2^{\sum_{a\in\mathcal{U}}n\mathbb{P}_U(a)(1-\delta)H(V|U=a)(1-\delta)}\\ &= (1-\delta)^{|\mathcal{U}|}2^{n(1-\delta)^2\sum_{a\in\mathcal{U}}\mathbb{P}_U(a)H(V|U=a)}\\ &= (1-\delta)^{|\mathcal{U}|}2^{n(1-2\delta+\delta^2)H(V|U)}\\ &\geq (1-\delta)^{|\mathcal{U}|}2^{n(1-2\delta)H(V|U)}. \end{aligned}$$

Hence, there are at least $(1-\delta)^{|\mathcal{U}|}2^{nH(V|U)(1-2\delta)}$ sequences $v^n\in\mathcal{V}^n$ satisfying (3). On the other hand, Part (a) shows that the condition (3) implies that $(u^n,v^n)\in\mathcal{T}_{3\delta}^n(p_{U,V})$. We conclude that there are at least $(1-\delta)^{|\mathcal{U}|}2^{nH(V|U)(1-2\delta)}$ sequences $v^n\in\mathcal{V}^n$ satisfying $(u^n,v^n)\in\mathcal{T}_{3\delta}^n(p_{U,V})$.

(d) If $(u^n,v^n)\in\mathcal{T}_{3\delta}^n(p_{U,V})$ then $v^n\in\mathcal{T}_{3\delta}^n(p_V)$ which implies that $\mathbb{P}_{V^n}(v^n)\geq 2^{-nH(V)(1+3\delta)}$.

(e) We have

$$\begin{aligned}
\mathbb{P}[(u^n, V^n) \in \mathcal{T}^n_{3\delta}(p_{U,V})] &= \sum_{\substack{v^n \in \mathcal{V}^n: \\ (u^n,v^n)\in\mathcal{T}^n_{3\delta}(p_{U,V})}} \mathbb{P}_{V^n}(v^n) \overset{(*)}{\geq} \sum_{\substack{v^n \in \mathcal{V}^n: \\ (u^n,v^n)\in\mathcal{T}^n_{3\delta}(p_{U,V})}} 2^{-nH(V)(1+3\delta)} \\
&= \left|\left\{v^n \in \mathcal{V}^n : \ (u^n, v^n) \in \mathcal{T}^n_{3\delta}(p_{U,V})\right\}\right| \cdot 2^{-nH(V)(1+3\delta)} \\
&\overset{(**)}{\geq} (1-\delta)^{|\mathcal{U}|} 2^{nH(V|U)(1-2\delta)} 2^{-nH(V)(1+3\delta)} \\
&= (1-\delta)^{|\mathcal{U}|} 2^{nH(V|U) - 2\delta H(V|U) - nH(V) - 3\delta H(V)} \\
&\overset{(***)}{\geq} (1-\delta)^{|\mathcal{U}|} 2^{-nI(U;V) - 2\delta \log|\mathcal{V}| - 3\delta \log|\mathcal{V}|} \\
&= (1-\delta)^{|\mathcal{U}|} 2^{-n[I(U;V) + 5\delta \log|\mathcal{V}|]},
\end{aligned}$$

where $(*)$ follows from Part (d), $(**)$ follows from Part (c) and $(***)$ follows from the fact that $I(U;V) = H(V) - H(V|U)$ and $H(V|U) \leq H(V) \leq \log|\mathcal{V}|$.