ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 7 | Information Theory and Coding

Solutions to homework 3 | Oct. 07, 2014

PROBLEM 1.

(a) $H(X) = \frac{2}{3}\log\frac{3}{2} + \frac{1}{3}\log 3 = 0.918$ bits $= H(Y)$.

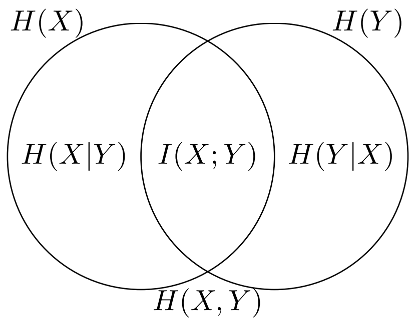
(b) $H(X|Y) = \frac{1}{3}H(X|Y=0) + \frac{2}{3}H(X|Y=1) = 0.667$ bits $= H(Y|X)$.

(c) $H(X,Y) = 3 \times \frac{1}{3}\log 3 = 1.585$ bits.

(d) $H(Y) - H(Y|X) = 0.251$ bits.

(d) $I(X;Y) = H(Y) - H(Y|X) = 0.251$ bits.

(f)



PROBLEM 2.

$$
\begin{aligned}
H(X) &= -\sum_{k=1}^{M} P_X(a_k)\log P_X(a_k) \\
&= -\sum_{k=1}^{M-1} (1-\alpha)P_Y(a_k)\log[(1-\alpha)P_Y(a_k)] - \alpha\log\alpha \\
&= (1-\alpha)H(Y) - (1-\alpha)\log(1-\alpha) - \alpha\log\alpha
\end{aligned}
$$

Since Y is a random variable that takes $M-1$ values $H(Y) \le \log(M-1)$ with equality if and only if Y takes each of its possible values with equal probability.

PROBLEM 3.

(a) Using the chain rule for mutual information,

$$I(X,Y;Z) = I(X;Z) + I(Y;Z \mid X) \ge I(X;Z),$$

with equality iff $I(Y;Z \mid X) = 0$, that is, when Y and Z are conditionally independent given X.

(b) Using the chain rule for conditional entropy,

$$H(X,Y \mid Z) = H(X \mid Z) + H(Y \mid X,Z) \ge H(X \mid Z),$$

with equality iff $H(Y \mid X,Z) = 0$, that is, when Y is a function of X and Z.

(c) Using first the chain rule for entropy and then the definition of conditional mutual information,

$$\begin{aligned} H(X,Y,Z) - H(X,Y) &= H(Z \mid X,Y) = H(Z \mid X) - I(Y;Z \mid X) \\ &\le H(Z \mid X) = H(X,Z) - H(X)\,, \end{aligned}$$

with equality iff $I(Y;Z \mid X) = 0$, that is, when Y and Z are conditionally independent given X.

(d) Using the chain rule for mutual information,

$$I(X;Z \mid Y) + I(Z;Y) = I(X,Y;Z) = I(Z;Y \mid X) + I(X;Z)\,,$$

and therefore

$$I(X;Z \mid Y) = I(Z;Y \mid X) - I(Z;Y) + I(X;Z)\,.$$

We see that this inequality is actually an equality in all cases.

Problem 4. Let X^i denote $X_1, \ldots, X_i$.

(a) By the chain rule for entropy,

$$\begin{aligned} \frac{H(X_1, X_2, \ldots, X_n)}{n} &= \frac{\sum_{i=1}^{n} H(X_i|X^{i-1})}{n} && (1) \\ &= \frac{H(X_n|X^{n-1}) + \sum_{i=1}^{n-1} H(X_i|X^{i-1})}{n} && (2) \\ &= \frac{H(X_n|X^{n-1}) + H(X_1, X_2, \ldots, X_{n-1})}{n}. && (3) \end{aligned}$$

From stationarity it follows that for all $1 \le i \le n$,

$$H(X_n|X^{n-1}) \le H(X_i|X^{i-1}),$$

which further implies, by summing both sides over $i = 1, \ldots, n-1$ and dividing by $n-1$, that,

$$\begin{aligned} H(X_n|X^{n-1}) &\le \frac{\sum_{i=1}^{n-1} H(X_i|X^{i-1})}{n-1} && (4) \\ &= \frac{H(X_1, X_2, \ldots, X_{n-1})}{n-1}. && (5) \end{aligned}$$

Combining (3) and (5) yields,

$$\begin{aligned} \frac{H(X_1, X_2, \ldots, X_n)}{n} &\le \frac{1}{n}\left[\frac{H(X_1, X_2, \ldots, X_{n-1})}{n-1} + H(X_1, X_2, \ldots, X_{n-1})\right] && (6) \\ &= \frac{H(X_1, X_2, \ldots, X_{n-1})}{n-1}. && (7) \end{aligned}$$

(b) By stationarity we have for all $1 \le i \le n$,

$$H(X_n|X^{n-1}) \le H(X_i|X^{i-1}),$$

which implies that,

$$\begin{aligned} H(X_n|X^{n-1}) &= \frac{\sum_{i=1}^{n} H(X_n|X^{n-1})}{n} && (8) \\ &\le \frac{\sum_{i=1}^{n} H(X_i|X^{i-1})}{n} && (9) \\ &= \frac{H(X_1, X_2, \ldots, X_n)}{n}. && (10) \end{aligned}$$

Problem 5. By the chain rule for entropy,

$$\begin{aligned} H(X_0|X_{-1},\dots,X_{-n}) &= H(X_0,X_{-1},\dots,X_{-n}) - H(X_{-1},\dots,X_{-n}) && (11)\\ &= H(X_0,X_1,\dots,X_n) - H(X_1,\dots,X_n) && (12)\\ &= H(X_0|X_1,\dots,X_n), && (13) \end{aligned}$$

where (12) follows from stationarity.

Problem 6. For a Markov chain, given X_0 and X_n are independent given X_{n-1}. Thus

$$H(X_0|X_nX_{n-1}) = H(X_0|X_{n-1})$$

But, since conditioning reduces entropy,

$$H(X_0|X_nX_{n-1}) \leq H(X_0|X_n).$$

Putting the above together we see that $H(X_0|X_{n-1}) \leq H(X_0|X_n)$.