Problem 1. Let $W : \{0, 1\} \rightarrow \mathcal{Y}$ be a channel where the input is binary and where the output alphabet is $\mathcal{Y}$. The Bhattacharyya parameter of the channel $W$ is defined as

$$Z(W) = \sum_{y \in \mathcal{Y}} \sqrt{W(y|0)W(y|1)}.$$ 

Let $X_1, X_2$ be two independent random variables uniformly distributed in $\{0, 1\}$ and let $Y_1$ and $Y_2$ be the output of the channel $W$ when the input is $X_1$ and $X_2$ respectively, i.e., $P_{Y_1,Y_2|X_1,X_2}(y_1,y_2|x_1,x_2) = W(y_1|x_1)W(y_2|x_2)$. Define the channels $W^- : \{0, 1\} \rightarrow \mathcal{Y}^2$ and $W^+ : \{0, 1\} \rightarrow \mathcal{Y}^2 \times \{0, 1\}$ as follows:

- $W^-(y_1, y_2|u_1) = P[Y_1 = y_1, Y_2 = y_2|X_1 \oplus X_2 = u_1]$ for every $u_1 \in \{0, 1\}$ and every $y_1, y_2 \in \mathcal{Y}$, where $\oplus$ is the XOR operation.
- $W^+(y_1, y_2, u_1|u_2) = P[Y_1 = y_1, Y_2 = y_2, X_1 \oplus X_2 = u_1|X_2 = u_2]$ for every $u_1, u_2 \in \{0, 1\}$ and every $y_1, y_2 \in \mathcal{Y}$.

(a) Show that $W^-(y_1, y_2|u_1) = \frac{1}{2} \sum_{u_2 \in \{0, 1\}} W(y_1|u_1 \oplus u_2)W(y_2|u_2)$.

(b) Show that $W^+(y_1, y_2, u_1|u_2) = \frac{1}{2} W(y_1|u_1 \oplus u_2)W(y_2|u_2)$.

(c) Show that $Z(W^+) = Z(W)^2$.

For every $y \in \mathcal{Y}$ define $\alpha(y) = W(y|0)$, $\beta(y) = W(y|1)$ and $\gamma(y) = \sqrt{\alpha(y)\beta(y)}$.

(d) Show that

$$Z(W^-) = \sum_{y_1,y_2 \in \mathcal{Y}} \frac{1}{2} \sqrt{\left(\alpha(y_1)\alpha(y_2) + \beta(y_1)\beta(y_2)\right)\left(\alpha(y_1)\beta(y_2) + \beta(y_1)\alpha(y_2)\right)}.$$ 

(e) Show that for every $x, y, z, t \geq 0$ we have $\sqrt{x+y+z+t} \leq \sqrt{x} + \sqrt{y} + \sqrt{z} + \sqrt{t}$. Deduce that

$$Z(W^-) \leq \frac{1}{2} \left( \sum_{y_1,y_2 \in \mathcal{Y}} \alpha(y_1)\gamma(y_2) \right) + \frac{1}{2} \left( \sum_{y_1,y_2 \in \mathcal{Y}} \alpha(y_2)\gamma(y_1) \right) + \frac{1}{2} \left( \sum_{y_1,y_2 \in \mathcal{Y}} \beta(y_1)\gamma(y_2) \right).$$

(f) Show that every sum in (1) is equal to $Z(W)$. Deduce that $Z(W^-) \leq 2Z(W)$.
**Problem 2.** For a given value $0 \leq z_0 \leq 1$, define the following random process:

\[ Z_0 = z_0, \quad Z_{i+1} = \begin{cases} Z_i^2 & \text{with probability } 1/2 \\ 2Z_i - Z_i^2 & \text{with probability } 1/2 \end{cases} \quad i \geq 0, \]

with the sequence of random choices made independently. Observe that the $Z$ process keeps track of the polarization of a Binary Erasure Channel with erasure probability $z_0$ as it is transformed by the polar transform: $P(Z_i = z)$ is exactly the fraction of Binary Erasure Channels having an erasure probability $z$ among the $2^i$ BEC channels which are synthesized by the polar transform at the $i$th level. The aim of this problem is to prove that for any $\delta > 0$, $\mathbb{P}[Z_i \in (\delta, 1 - \delta)] \to 0$ as $i$ gets large.

(a) Define $Q_i = \sqrt{Z_i(1 - Z_i)}$. Find $f_1(z)$ and $f_2(z)$ so that

\[ Q_{i+1} = Q_i \times \begin{cases} f_1(Z_i) & \text{with probability } 1/2, \\ f_2(Z_i) & \text{with probability } 1/2. \end{cases} \]

(b) Show that $f_1(z) + f_2(z) \leq \sqrt{3}$. Based on this, find a $\rho < 1$ so that

\[ \mathbb{E}[Q_{i+1} | Z_0, \ldots, Z_i] \leq \rho Q_i. \]

(c) Show that, for the $\rho$ you found in (b), $\mathbb{E}[Q_i] \leq \frac{1}{2} \rho^i$.

(d) Show that

\[ \mathbb{P}[Z_i \in (\delta, 1 - \delta)] = \mathbb{P}[Q_i > \sqrt{\delta(1 - \delta)}] \leq \frac{\rho^i}{2\sqrt{\delta(1 - \delta)}}. \]

Deduce that $\mathbb{P}[Z_i \in (\delta, 1 - \delta)] \to 0$ as $i$ gets large.

**Problem 3.** Let $P_1$ and $P_2$ be two channels of input alphabet $X_1$ and $X_2$ and of output alphabet $Y_1$ and $Y_2$ respectively. Consider a communication scheme where the transmitter chooses the channel ($P_1$ or $P_2$) to be used and where the receiver knows which channel were used. This scheme can be formalized by the channel $P$ of input alphabet $X = (X_1 \times \{1\}) \cup (X_2 \times \{2\})$ and of output alphabet $Y = (Y_1 \times \{1\}) \cup (Y_2 \times \{2\})$, which is defined as follows:

\[ P(y, k'|x, k) = \begin{cases} P_k(y|x) & \text{if } k' = k, \\ 0 & \text{otherwise}. \end{cases} \]

Let $X = (X_k, K)$ be a random variable in $X$ which will be the input distribution to the channel $P$, and let $Y = (Y_k, K) \in Y$ be the output distribution. Define $X_1$ as being the random variable in $X_1$ obtained by conditioning $X_k$ on $K = 1$. Similarly define $X_2, Y_1$ and $Y_2$. Let $\alpha$ be the probability that $K = 1$.

(a) Show that $I(X; Y) = h_2(\alpha) + \alpha I(X_1; Y_1) + (1 - \alpha)I(X_2; Y_2)$.

(b) What is the input distribution $X$ that achieves the capacity of $P$?

(c) Show that the capacity $C$ of $P$ satisfies $2^C = 2^{C_1} + 2^{C_2}$, where $C_1$ and $C_2$ are the capacities of $P_1$ and $P_2$ respectively.