**Problem 1.** (20 pts)

(a) (10 pts) Let \( U \) be a random variable taking values in the alphabet \( \mathcal{U} \), and let \( f \) be a mapping from \( \mathcal{U} \) to \( \mathcal{V} \). Show that \( H(f(U)) \leq H(U) \).

(b) (10 pts) Let \( U \) and \( V \) be two random variables taking values in the alphabets \( \mathcal{U} \) and \( \mathcal{V} \) respectively, and let \( f \) be a mapping from \( \mathcal{V} \) to \( \mathcal{W} \). Show that \( H(U|V) \leq H(U|f(V)) \).

**Problem 2.** (15 pts)

(a) (10 pts) Let \( U \) and \( \hat{U} \) be two random variables taking values in the same alphabet \( \mathcal{U} \), and let \( p_e = \mathbb{P}[U \neq \hat{U}] \). Show that \( H(U|\hat{U}) \leq h(p_e) + p_e \log(|\mathcal{U}| - 1) \), where \( h(p) = p \log \frac{1}{p} + (1-p) \log \frac{1}{1-p} \). Hint: use the random variable \( W \in \{0, 1\} \) defined by

\[
W = \begin{cases} 
1 & \text{if } U \neq \hat{U}, \\
0 & \text{otherwise}.
\end{cases}
\]

(b) (5 pts) Let \( U \) and \( V \) be two random variables taking values in the alphabets \( \mathcal{U} \) and \( \mathcal{V} \) respectively, and let \( f \) be a mapping from \( \mathcal{V} \) to \( \mathcal{W} \). Define \( p_e = \mathbb{P}[U \neq f(V)] \). Show that \( H(U|V) \leq h(p_e) + p_e \log(|\mathcal{U}| - 1) \).

**Problem 3.** (20 pts) Let \( U \) and \( W \) be two independent random variables uniformly distributed in \( \{0, 1\} \), and \( W' = U \oplus W \). Moreover, \( Z \) and \( Z' \) are two independent Bernoulli(\( p \)) random variables in \( \{0, 1\} \), where \( p \in (0, \frac{1}{2}) \). We also assume that \( Z \) and \( Z' \) are independent from \( U \) and \( W \). Define \( V = U \oplus Z \) and \( V' = U \oplus Z' \).

(a) (5 pts) Show that \( U \indep V \indep W \) and \( U \indep V' \indep W' \) are Markov chains. Deduce that \( I(U; V) \geq I(U; W) \) and \( I(U; V') \geq I(U; W') \).

(b) (5 pts) Compute \( I(U; V), I(U; W), I(U; V') \) and \( I(U; W') \). Deduce that \( I(U; V) > I(U; W) \) and \( I(U; V') > I(U; W') \).

(c) (10 pts) Compute \( I(U; VV') \) and \( I(U; WW') \). Deduce that \( I(U; VV') < I(U; WW') \).

**Problem 4.** (15 pts)

(a) (10 pts) Show that for every \( p, q \geq 0 \), we have

\[
\frac{1}{2} \left( p \log \frac{1}{p} + q \log \frac{1}{q} \right) \leq \frac{p+q}{2} \log \frac{2}{p+q}.
\]

(b) (5 pts) The entropy \( H(U) \) of a random variable \( U \) is a function of the distribution \( p_U \) of the random variable. Denote by \( h(p) \) the entropy of a random variable with distribution \( p \), i.e., \( h(p) = \sum_{u \in \mathcal{U}} p(u) \log \frac{1}{p(u)} \). Let \( p \) and \( q \) be two probability distributions on the same alphabet \( \mathcal{U} \), and let \( r \) be the probability distribution on \( \mathcal{U} \) defined by \( r(u) = \frac{p(u) + q(u)}{2} \) for every \( u \in \mathcal{U} \). Show that \( H(r) \geq \frac{1}{2} H(p) + \frac{1}{2} H(q) \).
Problem 5. (30 pts) Consider a source $U$ with alphabet $\mathcal{U}$ and suppose that we know that the true distribution of $U$ is either $P_1$ or $P_2$. Define $S = \sum_{u \in \mathcal{U}} \max\{P_1(u), P_2(u)\}$.

(a) (10 pts) Show that $S \leq 2$ and give a necessary and sufficient condition for equality.

(b) (5 pts) Show that there exists a prefix-free code where the length of the codeword associated to each symbol $u \in \mathcal{U}$ is $l(u) = \left\lceil \log_2 \frac{S}{\max\{P_1(u), P_2(u)\}} \right\rceil$.

(c) (5 pts) Show that the average length $\bar{l}$ (using the true distribution) of the code constructed in (b) satisfies $H(U) \leq \bar{l} < H(U) + \log S + 1 \leq H(U) + 2$.

Now assume that the true distribution of $U$ is one of $k$ distributions $P_1, \ldots, P_k$.

(d) (10 pts) Show that there exists a prefix-free code satisfying $H(U) \leq \bar{l} < H(U) + \log_2 S + 1 \leq H(U) + \log_2 k + 1$, where $S = \sum_{u \in \mathcal{U}} \max\{P_1(u), \ldots, P_k(u)\}$. 
