Problem 1. Let $p_{XY}(x,y)$ be given by
\[
\begin{array}{c|cc}
X \hspace{1cm} Y \\
0 & 1/3 & 1/3 \\
1 & 0 & 1/3 \\
\end{array}
\]
Find
(a) $H(X)$, $H(Y)$.
(b) $H(X|Y)$, $H(Y|X)$.
(c) $H(X,Y)$.
(d) $H(Y) - H(Y|X)$.
(e) $I(X;Y)$.
(f) Draw a Venn diagram for the quantities in (a) through (e).

Problem 2. Let $X$ be a random variable taking values in $M$ points $a_1, \ldots , a_M$, and let $P_X(a_M) = \alpha$. Show that
\[
H(X) = \alpha \log \frac{1}{\alpha} + (1 - \alpha) \log \frac{1}{1 - \alpha} + (1 - \alpha)H(Y)
\]
where $Y$ is a random variable taking values in $M - 1$ points $a_1, \ldots , a_{M-1}$ with probabilities $P_Y(a_j) = P_X(a_j)/(1-\alpha)$; $1 \leq j \leq M - 1$. Show that
\[
H(X) \leq \alpha \log \frac{1}{\alpha} + (1 - \alpha) \log \frac{1}{1 - \alpha} + (1 - \alpha) \log(M - 1)
\]
and determine the condition for equality.

Problem 3. Let $X, Y, Z$ be discrete random variables. Prove the validity of the following inequalities and find the conditions for equality:
(a) $I(X,Y;Z) \geq I(X;Z)$.
(b) $H(X,Y|Z) \geq H(X|Z)$.
(c) $H(X,Y,Z) - H(X,Y) \leq H(X,Z) - H(X)$.
(d) $I(X;Z|Y) \geq I(Z;Y|X) - I(Z;Y) + I(X;Z)$. 
PROBLEM 4. For a stationary process $X_1, X_2, \ldots$, show that

(a) $\frac{1}{n} H(X_1, \ldots, X_n) \leq \frac{1}{n-1} H(X_1, \ldots, X_{n-1})$.

(b) $\frac{1}{n} H(X_1, \ldots, X_n) \geq H(X_n|X_{n-1}, \ldots, X_1)$.

PROBLEM 5. Let $\{X_i\}_{i=-\infty}^{\infty}$ be a stationary stochastic process. Prove that

$$H(X_0|X_{-1}, \ldots, X_{-n}) = H(X_0|X_1, \ldots, X_n).$$

That is: the conditional entropy of the present given the past is equal to the conditional entropy of the present given the future.

PROBLEM 6. Show, for a Markov chain, that

$$H(X_0|X_n) \geq H(X_0|X_{n-1}), \quad n \geq 1.$$

Thus, initial state $X_0$ becomes more difficult to recover as time goes by.