Problem 1. Let $p_{XY}(x,y)$ be given by

<table>
<thead>
<tr>
<th>$X$</th>
<th>$Y$</th>
<th>0</th>
<th>1</th>
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<tbody>
<tr>
<td>0</td>
<td>1/3</td>
<td>1/3</td>
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<tr>
<td>1</td>
<td>0</td>
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</table>

Find

(a) $H(X), H(Y)$.

(b) $H(X|Y), H(Y|X)$.

(c) $H(X,Y)$.

(d) $H(Y) - H(Y|X)$.

(e) $I(X;Y)$.

(f) Draw a Venn diagram for the quantities in (a) through (e).

Problem 2. Let $X$ be a random variable taking values in $M$ points $a_1, \ldots, a_M$, and let $P_X(a_M) = \alpha$. Show that

$$H(X) = \alpha \log \frac{1}{\alpha} + (1 - \alpha) \log \frac{1}{1 - \alpha} + (1 - \alpha) H(Y)$$

where $Y$ is a random variable taking values in $M-1$ points $a_1, \ldots, a_{M-1}$ with probabilities $P_Y(a_j) = P_X(a_j)/(1 - \alpha); 1 \leq j \leq M - 1$. Show that

$$H(X) \leq \alpha \log \frac{1}{\alpha} + (1 - \alpha) \log \frac{1}{1 - \alpha} + (1 - \alpha) \log(M - 1)$$

and determine the condition for equality.

Problem 3. Let $X, Y, Z$ be discrete random variables. Prove the validity of the following inequalities and find the conditions for equality:

(a) $I(X, Y; Z) \geq I(X; Z)$.

(b) $H(X, Y|Z) \geq H(X|Z)$.

(c) $H(X, Y, Z) - H(X, Y) \leq H(X, Z) - H(X)$.

(d) $I(X; Z|Y) \geq I(Z; Y|X) - I(Z; Y) + I(X; Z)$.
Problem 4. For a stationary process $X_1, X_2, \ldots$, show that

(a) $\frac{1}{n} H(X_1, \ldots, X_n) \leq \frac{1}{n-1} H(X_1, \ldots, X_{n-1}).$

(b) $\frac{1}{n} H(X_1, \ldots, X_n) \geq H(X_n | X_{n-1}, \ldots, X_1).$

Problem 5. Let $\{X_i\}_{i=-\infty}^{\infty}$ be a stationary stochastic process. Prove that

$$H(X_0 | X_{-1}, \ldots, X_{-n}) = H(X_0 | X_1, \ldots, X_n).$$

That is: the conditional entropy of the present given the past is equal to the conditional entropy of the present given the future.

Problem 6. Show, for a Markov chain, that

$$H(X_0 | X_n) \geq H(X_0 | X_{n-1}), \quad n \geq 1.$$

Thus, initial state $X_0$ becomes more difficult to recover as time goes by.