on to Chapter 8 / 7:

a goody bag of assorted fancy counting tricks, without proofs
Counting using recurrence relations

section 8.1 / 7.1:
examples of non-obvious counting problems that allow easy reduction to sub-problems

general approach:
• solution $a_n$ to problem of size $n$
  is written as function $f$ of $a_1,a_2,\ldots,a_{n-1}$
• depending on $f$ this may (or may not) lead to a way to determine $a_n$ (in later sections)

elements: runtimes from earlier sections
• binary search among $n$ items in $b_n = b_{n/2} + C$
• mergesort of $n$ items in $m_n = 2m_{n/2} + n$, solved using ad hoc techniques and MI
Counting examples, section 8.1 / 7.1

compound interest:

• deposit $d_0 = x$ at 2% interest, $d_n$ after $n$ years: clearly $d_n = 1.02d_{n-1}$ and thus $d_n = 1.02^n x$

• with additional annual contribution of $y$:
  $d_n = 1.02d_{n-1} + y$, general solution more work

(undying) rabbits, or $\#$ $n$-bitstrings without “00”:

$a_n = a_{n-1} + a_{n-2}$ (Fibonacci, different $a_1, a_2$)

towers of Hanoi: $h_1 = 1, h_n = 2h_{n-1} + 1 = \ldots = 2^n - 1$

$\#$ $n$-digit integers with even number of zeros:

$a_n = 9a_{n-1} + (10^{n-1} - a_{n-1}) = 8a_{n-1} + 10^{n-1}$

$\#$ parenthizizations of $x_0 x_1 \ldots x_n$ (Catalan):

$C_n = C_0 C_{n-1} + C_1 C_{n-2} + \ldots + C_{n-2} C_1 + C_{n-1} C_0$
8.2 / 7.2: Solving (some of) these recurrences

Solving \( a_n = c_1 a_{n-1} \) was easy: \( a_n = (c_1)^n a_0 \)

Next case, \( a_n = c_1 a_{n-1} + c_2 a_{n-2} \), is harder:

is degree 2 case of

linear homogeneous recurrence relation of degree \( k \):

\[
a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}
\]

where \( c_i \)'s are real constants and \( c_k \neq 0 \)

✓ \( d_n = 1.02 d_{n-1} \), of degree 1

• \( d_n = 1.02 d_{n-1} + y \): nonhomogeneous

✓ \( a_n = a_{n-1} + a_{n-2} \), of degree 2

• \( h_n = 2h_{n-1} + 1 \): nonhomogeneous

• \( m_n = 2m_{n/2} + n \): no fixed degree, nonhomogeneous

• \( C_n = C_0 C_{n-1} + C_1 C_{n-2} + \ldots + C_{n-1} C_0 \): nonlinear
Solving \( a_n = c_1 a_{n-1} + c_2 a_{n-2} \) \((c_2 \neq 0)\)

try \( a_n = r^n \) as solution (for unknown \( r \neq 0 \)):

\[
a_n = c_1 a_{n-1} + c_2 a_{n-2} \iff r^n = c_1 r^{n-1} + c_2 r^{n-2}
\]

\[
\iff r^n - c_1 r^{n-1} - c_2 r^{n-2} = 0
\]

\[
\iff r^2 - c_1 r - c_2 = 0
\]

\( \forall r: r^2 - c_1 r - c_2 = 0 \) and \( \alpha_r \in \mathbb{R} \):

\[
a_n = \Sigma_r \alpha_r r^n \text{ solves } a_n = c_1 a_{n-1} + c_2 a_{n-2}
\]

polynomial \( r^2 - c_1 r - c_2 \) has 2 or 1 roots:

- 2 roots: \( r_1, r_2 \) with \( r_1 \neq r_2 \), \( a_n = \alpha_1 r_1^n + \alpha_2 r_2^n \)
- single root \( r \): \( a_n = \alpha_1 r^n + \alpha_2 n r^n \) (root of derivative as well)

with \( \alpha_i \) determined by \( a_0 \) and \( a_1 \)

conversely: each solution of this form
Example: solving \( f_n = f_{n-1} + f_{n-2}, \ f_0 = 0, \ f_1 = 1 \)

\( r^2 - r - 1 \) has roots \( (1 \pm \sqrt{5})/2 \)

\[ \Rightarrow f_n = \alpha_1((1+\sqrt{5})/2)^n + \alpha_2((1-\sqrt{5})/2)^n \]

from \( f_0 = \alpha_1 + \alpha_2 = 0 \)

and \( f_1 = \alpha_1(1+\sqrt{5})/2 + \alpha_2(1-\sqrt{5})/2 = 1 \)

it follows that \( \alpha_1 = 1/\sqrt{5} \) and \( \alpha_2 = -1/\sqrt{5} \)

\[ \Rightarrow \text{the } n\text{th Fibonacci number is} \]

\[ f_n = \frac{( (1+\sqrt{5})^n - (1-\sqrt{5})^n )}{(2^n\sqrt{5})} \]
Example: solving $d_n = 4d_{n-1} - 4d_{n-2}, \ d_0 = d_1 = 1$

$r^2 - 4r + 4 = (r - 2)^2$ has double root 2

$\Rightarrow d_n = \alpha_1 2^n + \alpha_2 n 2^n$

from $d_0 = \alpha_1 = 1$

and $d_1 = \alpha_1 2 + \alpha_2 2 = 1$

it follows that $\alpha_1 = 1$ and $\alpha_2 = -\frac{1}{2}$

$\Rightarrow d_n = 2^n - n 2^{n-1}$
Solving \( a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} \) \((c_k \neq 0)\)

same approach: try \( a_n = r^n \) as solution …:

polynomial \( r^k - c_1 r^{k-1} - \ldots - c_k \) has \( \leq k \) roots:
- all distinct roots \( r_1, r_2, \ldots, r_k \):
  \[
  a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \ldots + \alpha_k r_k^n
  \]
- roots with multiplicities: more complicated with \( \alpha_i \) determined by \( a_0, a_1, \ldots, a_{k-1} \)

conversely: each solution of this form

Note:
- \( r^k - c_1 r^{k-1} - \ldots - c_k \): characteristic equation
- its roots the characteristic roots
Handling the nonhomogeneous case

\[ a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n) \]

with \( c_i \) s real constants, \( c_k \neq 0 \), and \( F(n) \neq 0 \): linear nonhomogeneous recurrence relation

with \( a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} \) as its associated homogeneous recurrence relation

- any solution to the latter (which we know) can be added to solution \( a_n^{(p)} \) to the former
- particular solution \( a_n^{(p)} \) always exists if

\[ F(n) = (\text{degree } t \text{ polynomial in } n) \times s^n: \]

\[ a_n^{(p)} = n^m \left( p_t n^t + p_{t-1} n^{t-1} + \ldots + p_0 \right) s^n \]

where \( p_t \neq 0 \) and \( m \) is multiplicity of \( s \) as root of \( r^k - c_1 r^{k-1} - \ldots - c_k \)
Example application “nonhomogeneous”

Hanoi, \( h_1 = 1, \ h_n = 2h_{n-1} + 1 = \ldots = 2^n - 1 \):

- Characteristic equation (“CE”) \( r - 2 = 0 \)
  \( \Rightarrow \) solution to homogeneous part is \( \alpha 2^n \)

- \( F(n) = 1 = (\text{degree } t \text{ polynomial in } n) \times s^n \)
  \( t = 0, \ s = 1 : \)
  \( \Rightarrow \ a_n^{(p)} = n^m (p_0) 1^n \) a particular solution
  \( s = 1 \) is not a root of CE, so \( m = 0 \)
  substitute \( a_n^{(p)} = p_0 \) in \( h_n = 2h_{n-1} + 1 \)
  \( \Rightarrow p_0 = -1 \)

- General solution of form \( \alpha 2^n - 1 \)
  use \( h_1 = 1 \) \( \Rightarrow \alpha 2^1 - 1 = 1 \) \( \Rightarrow \alpha = 1 \)

- Solution is \( 2^n - 1 \)
Another application of “nonhomogeneous” interest, \( d_0 = x, d_n = 1.02 d_{n-1} + y \), where \( y \neq 0 \):

- characteristic equation ("CE") \( r - 1.02 = 0 \)
  \( \Rightarrow \) solution to homogeneous part is \( \alpha 1.02^n \)

- \( F(n) = y = (\text{degree } t \text{ polynomial in } n) \times s^n \)
  \( t = 0, s = 1 \):
  \( \Rightarrow a_n^{(p)} = n^m (p_0) l^n \) a particular solution
  \( s = 1 \) is not a root of CE, so \( m = 0 \)
  substitute \( a_n^{(p)} = p_0 \) in \( d_n = 1.02 d_{n-1} + y \)
  \( \Rightarrow p_0 = -y/0.02 \)

- general solution of form \( \alpha 1.02^n - y/0.02 \)
  use \( d_0 = x \) \( \Rightarrow \alpha 1.02^0 - y/0.02 = x \) \( \Rightarrow \alpha = x + y/0.02 \)

- solution is \( (x + y/0.02)1.02^n - y/0.02 \)
Another example

$a_n$: the number of length $n$ ternary strings (0s, 1s, 2s) with an even number of 1s

$n = 0$: "" is unique empty string, no 1s: $a_0 = 1$

$n = 1$: “0” and “2”, thus $a_1 = 2$

$n = 2$: “00”, “02”, “11”, “20”, “22”, thus $a_2 = 5$

recurrence relation?

to get a proper length $n+1$ string:

• take any (of $3^n$) ternary length $n$ string, add a “0” (if even 1s) or “1” (if odd 1s)

• or add “2” to any of the $a_n$ length $n$ strings

$\Rightarrow a_{n+1} = a_n + 3^n$, example of nonhomogeneous

(& confirming that $a_0 = 1$)
Solving nonhomogeneous $a_{n+1} = a_n + 3^n$
(with $a_0 = 1$, $a_1 = 2$, $a_2 = 5$)

characteristic equation ("CE") $r - 1 = 0$

$\Rightarrow$ Solution to homogeneous part is $\alpha 1^n = \alpha$

- $F(n) = 3^n = (\text{degree } t \text{ polynomial in } n) \times s^n$
  
  $t = 0, s = 3$: 
  
  $\Rightarrow a_n^{(p)} = n^m (p_0) 3^n$ a particular solution
  
  $s = 3$ is not a root of CE, so $m = 0$

  substitute $a_n^{(p)} = p_0 3^n$ in $a_{n+1} = a_n + 3^n$
  
  $\Rightarrow p_0 3^{n+1} = p_0 3^n + 3^n \Rightarrow p_0 = \frac{1}{2}$

- general solution of form $\alpha + \frac{1}{2} 3^n$

  use $a_0 = 1 \Rightarrow \alpha + \frac{1}{2} = 1 \Rightarrow \alpha = \frac{1}{2}$

- solution is $\frac{1}{2}(1 + 3^n)$ (is intuitively about right)
Final section 8.2 / 7.2 example application

sum of squares $a_n = \sum_{i=1}^{n} i^2 \Rightarrow a_n = a_{n-1} + n^2$

• characteristic equation ("CE") $r - 1 = 0$

$\Rightarrow$ solution to homogeneous part is $\alpha 1^n = \alpha$

• $F(n) = n^2 = (\text{degree 2 polynomial in } n) \times s^n$

$t = 2, s = 1 \Rightarrow a_n^{(p)} = n^m (p_2 n^2 + p_1 n + p_0) 1^n$

is a particular solution; since $s = 1$ is a root of CE of multiplicity 1 it follows that $m = 1$

• substitute $a_n^{(p)} = n(p_2 n^2 + p_1 n + p_0)$

in $a_n = a_{n-1} + n^2$

and solve for $p_0, p_1, p_2$ using $a_0, a_1, a_2$

(very much like we’ve determined $a_n$ before)

• finally use solution $a_n^{(p)} + \alpha$ to conclude $\alpha = 0$
Brief remark on 8.3 / 7.3: simple tricks
so far: linear (non)homogeneous recurrences
of fixed degree
not suitable to solve
• \( b_n = b_{n/2} + C \) (binary search runtime)
  \[ b_n = O(\log n) \]
• \( m_n = 2m_{n/2} + n \) (mergesort runtime)
  \[ m_n = O(n \log n) \]
• \( k_n = 3k_{n/2} + Cn \) (Karatsuba runtime)
  \[ k_n = O(n^{\log_2 3}) \]
• \( s_n = 7s_{n/2} + Cn^2 \) (Strassen’s Karatsuba-like matrix ×)
  \[ s_n = O(n^{\log_2 7}) \]
• etc: see: thms 1&2, pages 514&516 / 477&479
  (or use common sense)
8.4 / 7.4 Generating functions

solving counting problems
by interpreting coefficients of polynomials
(or power series) as the required solutions

simple example:
number of non-negative integer solutions to
\[ e_1 + e_2 + e_3 = 4 \]

• pick 4 cookies from 3 types of cookies in
  \[ 3+4-1 \text{ choose } 3-1 = 15 \text{ ways} \]

• pick \( x^{e_1} \) from \( 1 + x + x^2 + x^3 + ... = 1/(1-x) \),
  pick \( x^{e_2} \) from \( 1 + x + x^2 + x^3 + ... = 1/(1-x) \), and
  pick \( x^{e_3} \) from \( 1 + x + x^2 + x^3 + ... = 1/(1-x) \)

\[ \Rightarrow \text{need coefficient of } x^4 \text{ in } 1/(1-x)^3 \]
Power series, basics

- power series is a polynomial of possibly infinite degree:
  \[ f(x) = \sum_{i=0}^{\infty} f_i x^i, \quad g(x) = \sum_{j=0}^{\infty} g_j x^j \]

- define: \( f(x) + g(x) = \sum_{i=0}^{\infty} (f_i + g_i) x^i \)
  \[ f(x)g(x) = \sum_{i=0}^{\infty} \left( \sum_{k=0}^{i} f_k g_{i-k} \right) x^i \]

- 1-to-1 correspondence between \( h \) and its Taylor series expansion (around \( a \)):
  \[ h(x) = \sum_{n=0}^{\infty} \frac{h^{(n)}(a)}{n!} (x - a)^n \quad (h^{(n)} : nth derivative) \]
Common power/Taylor series expansions

\[(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k\]

\[\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k\]

\[\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (k+1) x^k\]

\[\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k\]

\[e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}\]

\[\ln(1 + x) = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{x^k}{k}\]

plus common substitutions \((-x\text{ or } cx\text{ for } x)\), and term by term differentiation and integration
Back to simple example

number of non-negative integer solutions to

\[ e_1 + e_2 + e_3 = 4 \]

- pick 4 cookies from 3 types of cookies in
  \(3+4-1\) choose \(3-1 = 15\) ways
- pick \(x^{e_1}\) from \(1 + x + x^2 + x^3 + ... = \frac{1}{1-x}\),
  pick \(x^{e_2}\) from \(1 + x + x^2 + x^3 + ... = \frac{1}{1-x}\), and
  pick \(x^{e_3}\) from \(1 + x + x^2 + x^3 + ... = \frac{1}{1-x}\)

\[ \implies \text{need coefficient of } x^4 \text{ in } \frac{1}{(1-x)^3} \]

with

\[
\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k
\]

this coefficient equals \(\binom{3+4-1}{4} = 15\)
Another simple example

number of non-negative integer solutions to

\[ e_1 + e_2 + e_3 = 4 \]

such that \( e_2 \) is even and \( e_3 \) a multiple of 3

• unclear how to use basic earlier method
  • pick \( x^{e_1} \) from \( 1 + x + x^2 + x^3 + \ldots = 1/(1-x) \),
  • pick \( x^{e_2} \) from \( 1 + x^2 + x^4 + \ldots = 1/(1-x^2) \), and
  • pick \( x^{e_3} \) from \( 1 + x^3 + x^6 + \ldots = 1/(1-x^3) \)

\[ \Rightarrow \text{need coefficient of } x^4 \text{ in } 1/((1-x)(1-x^2)(1-x^3)) \]
Final simple example

number of non-negative integer solutions to

\[ e_1 + e_2 + e_3 + e_4 = 20 \]

with \( e_1 \) even, \( e_2 \) multiple of 5, \( e_3 \leq 4 \), \( e_4 \leq 1 \),
\[ x^{e_1} \text{ from } 1 + x^2 + x^4 + \ldots = 1/(1 - x^2), \text{ and} \]
\[ x^{e_2} \text{ from } 1 + x^5 + x^{10} + \ldots = 1/(1 - x^5) \]
\[ x^{e_3} \text{ from } 1 + x + x^2 + x^3 + x^4 = (1 - x^5)/(1 - x) \]
\[ x^{e_4} \text{ from } 1 + x = (1 - x^2)/(1 - x) \]

\[ \Rightarrow \text{ solution is } 21: \text{ the coefficient of } x^{20} \text{ in} \]

\[
\left( \frac{1}{1 - x^2} \right) \left( \frac{1}{1 - x^5} \right) \left( \frac{1 - x^5}{1 - x} \right) \left( \frac{1 - x^2}{1 - x} \right) = \frac{1}{(1 - x)^2}
\]
Generating function to solve recurrences

let \( a_0 = 1, \ a_n = 2a_{n-1} \)

show that \( a_n = 2^n \) using a generating function:

\[
A(x) = \sum_{i=0}^{\infty} a_i x^i \Rightarrow xA(x) = \sum_{j=1}^{\infty} a_{j-1} x^j
\]

\[
\Rightarrow A(x) - 2x A(x) = a_0 + \sum_{i=1}^{\infty} (a_i - 2a_{i-1}) x^i = 1
\]

\[
\Rightarrow A(x) = 1/(1 - 2x)
\]

we know that \( \sum_{i=0}^{\infty} r^i = 1/(1 - r) \) (for \( |r| \leq 1 \))

with \( r = 2x \) we find \( \sum_{i=0}^{\infty} (2x)^i = 1/(1 - 2x) \)

and thus \( \sum_{i=0}^{\infty} (2x)^i = A(x) \)

it follows that \( a_i = 2^i \)
Another example: \( a_n = a_{n-1} + n, \ a_0 = 0, \ a_1 = 1 \)

\[
A(x) = \sum_{i=0}^{\infty} a_i x^i = a_0 + \sum_{i=1}^{\infty} a_i x^i = \sum_{i=1}^{\infty} a_i x^i
\]

\[
\Rightarrow A(x) = \sum_{i=1}^{\infty} (a_{i-1} + i) x^i = \sum_{j=0}^{\infty} (a_j + j + 1) x^{j+1}
\]

\[
\Rightarrow A(x) = \sum_{j=0}^{\infty} a_j x^{j+1} + \sum_{j=0}^{\infty} (j + 1) x^{j+1}
\]

\[
\Rightarrow A(x) = x \sum_{j=0}^{\infty} a_j x^j + x \sum_{j=0}^{\infty} (j + 1) x^j
\]

\[
\Rightarrow A(x) = x A(x) + \frac{x}{(1-x)^2}
\]

(← and ↓ use page 526/489)

\[
\Rightarrow A(x) = \frac{x}{(1-x)^3} = \sum_{i=0}^{\infty} C(3+i-1, i) x^{i+1}
\]

\[
\Rightarrow A(x) = \sum_{j=1}^{\infty} C(j+1, j-1) x^j \Rightarrow a_n = \frac{n(n+1)}{2}
\]
The approach:

• interpret sequence $a_n$ to be determined as coefficients of a power series of some $A$

• use the recurrence relation to derive an alternative expression $f$ for $A$

• find (using a table, using Taylor, ...) power series expansion for $f$:

$$ f(x) = \sum_{i=0}^{\infty} f_i x^i $$

• coefficients $f_i$ are closed expression for $a_i$

(many more details in section 8.4 / 7.4)
\[ a_{n+1} = a_n + 3^n \] with generating functions
\[ a_0 = 1, \ a_1 = 2, \ a_2 = 5 \]

\[ A(x) = \sum_{i=0}^{\infty} a_i x^i = a_0 + \sum_{i=1}^{\infty} a_i x^i \]

\[ \Rightarrow A(x) = 1 + \sum_{j=0}^{\infty} a_{j+1} x^{j+1} = 1 + x \sum_{j=0}^{\infty} a_{j+1} x^j \]

\[ \Rightarrow A(x) = 1 + x \sum_{j=0}^{\infty} a_j x^j + x \sum_{j=0}^{\infty} 3^j x^j \]

\[ \Rightarrow A(x) = 1 + xA(x) + \frac{x}{1-3x} \]

\[ \Rightarrow (1-x)A(x) = 1 + \frac{x}{1-3x} = \frac{1-2x}{1-3x} \]

\[ \Rightarrow A(x) = \frac{1-2x}{(1-x)(1-3x)} \]
Continuation

we have \( A(x) = \frac{1 - 2x}{(1 - x)(1 - 3x)} \)

write \( A(x) = \frac{u}{1 - x} + \frac{v}{1 - 3x} \)

thus \( u(1 - 3x) + v(1 - x) = 1 - 2x \)

implying that \( u + v = 1 \) and \( 3u + v = 2 \)

thus \( u = v = 1/2 \) \( \text{and} \) \( A(x) = \frac{1}{2} \left( \frac{1}{1 - x} + \frac{1}{1 - 3x} \right) \)

it follows that \( A(x) = \sum_{i=0}^{\infty} \frac{1}{2} (1^i + 3^i) x^i \)

and thus that \( a_n = \frac{1}{2} (1 + 3^n) \)

(always check correctness of \( a_0, a_1, a_2 \))
Catalan numbers

\[ C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1} \quad \text{with} \quad C_0 = C_1 = 1 \]

let \( G(x) = \sum_{n=0}^{\infty} C_n x^n \)

\[ \Rightarrow G(x)^2 = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} C_k C_{n-k} \right) x^n \]
\[ = \sum_{n=1}^{\infty} \left( \sum_{k=0}^{n-1} C_k C_{n-1-k} \right) x^{n-1} \]
\[ = \sum_{n=1}^{\infty} C_n x^{n-1} \]

\[ \Rightarrow xG(x)^2 = \sum_{n=1}^{\infty} C_n x^n = G(x) - C_0 \]

\[ \Rightarrow xG(x)^2 - G(x) + 1 = 0 \]

\[ \Rightarrow G(x) = \frac{1 \pm \sqrt{1 - 4x}}{2x} \]
Catalan numbers, continued

\[ G(x) = \frac{1 - \sqrt{1 - 4x}}{2x} \quad (+ - \text{choice is bad at zero}) \]

let \( xG(x) = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4x} = f(x) \)

\[ \Rightarrow f'(x) = (1 - 4x)^{-1/2} \]

we will see that \( (1 - 4x)^{-1/2} = \sum_{n=0}^{\infty} \binom{2n}{n} x^n \)

\[ \Rightarrow f'(x) = \sum_{n=0}^{\infty} \binom{2n}{n} x^n, \text{ term by term integration :} \]

\[ \Rightarrow f(x) = c + \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} x^{n+1} \]

\[ \Rightarrow c + \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} x^{n+1} = xG(x) = \sum_{n=0}^{\infty} C_n x^{n+1} \]

\[ \Rightarrow C_n = \frac{1}{n+1} \binom{2n}{n} \]
Extended binomial coefficients and theorem

$n, k$ integers $\geq 0$: \( \binom{n}{k} = \frac{n(n-1)...(n-k+1)}{k!} \)

define for real $u$ and integer $k > 0$:

\( \binom{u}{k} = \frac{u(u-1)...(u-k+1)}{k!} \) \text{ and } \binom{u}{0} = 1

then for any real $u$ and real $x$ with $|x| < 1$:

\( (1 + x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k \)

compare to binomial theorem (integer $n \geq 0$):

\( (1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \)
Back to Catalan numbers, \((1-4x)^{-1/2}\) from \((1+x)^u = \sum_{n=0}^\infty \binom{u}{n} x^n\) it follows that

\[
(1-4x)^{-1/2} = \sum_{n=0}^\infty \binom{-1/2}{n} (-4x)^n
\]

we will see that \(\binom{-1/2}{n} = \binom{2n}{n} \frac{1}{(-4)^n}\)

thus \((1-4x)^{-1/2} = \sum_{n=0}^\infty \frac{\binom{2n}{n}}{(-4)^n} (-4x)^n = \sum_{n=0}^\infty \binom{2n}{n} x^n\)
Final step: \((-1/2\)^n\)
for positive integer \(n\):

\[
(-1/2)^n = \frac{(-1/2)((-1/2) - 1)\ldots((-1/2) - n + 1)}{n!} \\
= \frac{(-1/2)(-3/2)(-5/2)\ldots(-(2n-1)/2)}{n!} \\
= (-1)^n \frac{1\cdot3\cdot5\ldots(2n-1)}{2^n n!} \cdot \frac{2\cdot4\cdot6\ldots(2n)}{2^n n!} \\
= (-1)^n \frac{1\cdot2\cdot3\ldots(2n)}{4^n n!n!}
\]

\[\Rightarrow (-1/2)^n = \binom{2n}{n} \frac{1}{(-4)^n}\]
8.5 & 8.6 / 7.5 & 7.6: Inclusion & Exclusion
covered in homeworks and at midterm