Chapter 6 / 5: Counting

general observation & recommendations:
• messing up while counting is hard to avoid (despite attempts to capture counting in “rules”)
• try smaller examples that keep essence
• check that answer makes sense (negative counts are usually incorrect)
• verify consistency between different ways to count the same
Prelude: counting using Chinese remaindering

to find the number $C$ of people

- let $p$ and $q$ be coprime integers with $pq > C$
- form groups of $p$ persons: find $C_p = C \mod p$
- form groups of $q$ persons: find $C_q = C \mod q$
- thus $C = C_p + kp$ for unknown integer $k$
- to determine $k$, note that $C_p + kp = C_q \mod q$
  thus $k = (C_q - C_p)/p \mod q$
- to calculate $k$ we need $s$ such that $sp = 1 \mod q$
  and thus $s = 1/p \mod q$ (and $k = s(C_q - C_p) \mod q$)
- finding $s$: with “$0*p = q \mod q$” and
  “$1*p = p \mod q$”, perform the Euclidean algorithm on right hand sides until it equals 1
Warm-up: two simple counting rules

$A$ and $B$ are two different tasks, with $n$ ways to do $A$ and $m$ ways to do $B$

two scenarios:

1. task $A$ must be done followed by task $B$
   
   product rule:
   
   $n$ times $m$ ways to do $A$ and then $B$

2. task $A$ or task $B$ must be done (not both)
   
   sum rule:
   
   $n$ plus $m$ ways to do $A$ or $B$

question:

how many ways to carry out each scenario?
Trivial example: pick two bits, a 1\textsuperscript{st} & a 2\textsuperscript{nd} how many ways to pick the two bits?

- **task \( A \):** pick 1\textsuperscript{st} bit; 2 ways to do so
- **task \( B \):** pick 2\textsuperscript{nd} bit; 2 ways to do so

\[ \Rightarrow \text{do task } A \text{ followed by task } B \]

thus \( 2 \times 2 = 4 \) ways to do \( A \) followed by \( B \)

other way to define the tasks:

- **task \( A \):** 1\textsuperscript{st} bit is 0; 2 ways to pick 2\textsuperscript{nd} bit
- **task \( B \):** 1\textsuperscript{st} bit is 1; 2 ways to pick 2\textsuperscript{nd} bit

\[ \Rightarrow \text{do task } A \text{ or task } B \]

thus \( 2 + 2 = 4 \) ways to do \( A \) or \( B \)

(works because \( A \) and \( B \) are disjoint)
Common pitfall of sum rule

In how many ways can one pick seven bits such that last bit is 1 or the first 3 bits are 0?

A: pick last bit as 1; product rule: 64 ways
B: pick first 3 bits as 0; product rule: 16 ways

⇒ (sum rule?) 64 + 16 = 80 ways to do A or B

Wrong because A and B are not disjoint:
- 8 of the ways under A have first 3 bits 0, or
- half the ways (i.e., 8) under B have last bit 1

⇒ either way, subtract 8 from 80: result 72

(remember principle of inclusion and exclusion: \(|A \cup B| = |A| + |B| - |A \cap B|\))

\(B':\) first 3 bits and last bit all zero, 8 total, is disjoint with A: \(|A| + |B'| = 72\)
The first three of six simple examples
consider strings of length 6 over \{a,b,c,\ldots,y,z\}

1. how many?
   26 choices for 1\textsuperscript{st}, 26 choices for 2\textsuperscript{nd},
   26 for 3\textsuperscript{rd}, \ldots, and 26 for 6\textsuperscript{th}
   \[ \Rightarrow \text{product rule: } 26^6 \]

2. how many begin with a vowel \{a,e,i,o,u\}? 
   5 choices for 1\textsuperscript{st}, 26 choices for 2\textsuperscript{nd} – 6\textsuperscript{th}
   \[ \Rightarrow \text{product rule: } 5*26^5 \]

3. how many begin and end with a vowel?
   5 choices for 1\textsuperscript{st} and 6\textsuperscript{th}, 26 for 2\textsuperscript{nd} – 5\textsuperscript{th}
   \[ \Rightarrow \text{product rule: } 5*26^4*5 = 5^2*26^4 \]
Fourth simple example
consider strings of length 6 over \{a,b,c,\ldots,y,z\}
4. how many begin or end with a vowel?
\[5 \times 26^5\] begin with vowel
\[26^5 \times 5\] end with vowel
\[\Rightarrow 2 \times 5 \times 26^5\] begin or end with vowel
but we counted “begin and end” twice
\[\Rightarrow 2 \times 5 \times 26^5 - 5^2 \times 26^4 = 235 \times 26^4\]

alternative calculation: complement of those
that begin and end with consonant
\[\Rightarrow 26^6 - 21^2 \times 26^4 = (26^2 - 21^2) \times 26^4\]

use: \((c_1 = \text{vowel } \vee c_6 = \text{vowel}) \equiv \neg (c_1 \neq \text{vowel } \land c_6 \neq \text{vowel})\)
\[\equiv \neg (c_1 = \text{consonant } \land c_6 = \text{consonant})\]
Fifth simple example
consider strings of length 6 over \{a,b,c,\ldots,y,z\}
5. how many begin or end with a vowel, but not a vowel at begin and end?
   “begin or end” was $2 \times 5 \times 26^5 - 5^2 \times 26^4$
   need to subtract “begin and end” again:
   $\Rightarrow 2 \times 5 \times 26^5 - 2 \times 5^2 \times 26^4 = 210 \times 26^4$
alternative calculation:
   begin vowel, end consonant: $5 \times 26^4 \times 21$
   begin consonant, end vowel: $21 \times 26^4 \times 5$
   these two possibilities are disjoint
   $\Rightarrow$ sum rule:
   $5 \times 26^4 \times 21 + 21 \times 26^4 \times 5 = 210 \times 26^4$
Last simple example
consider strings of length 6 over \{a,b,c,\ldots,y,z\}
6. how many have precisely one vowel?
  1\textsuperscript{st} vowel, others consonants: 5\times21^5
  2\textsuperscript{nd} vowel, others consonants: 21\times5\times21^4
  3\textsuperscript{rd} vowel, others consonants: 21^2\times5\times21^3
  …
  6\textsuperscript{th} vowel, others consonants: 21^5\times5
(all possibilities disjoint)
\Rightarrow sum rule: 6\times5\times21^5
Brief pigeonhole discussion

if $N$ items are distributed over fewer than $N$ bins, then there is a bin with at least two items ($N > 1$)

example: in any group of $\geq 2$ persons there are at least 2 who have the same number of friends in the group ("being friends" is "symmetric"):

persons $p_1, p_2, \ldots, p_n$; $f(i)$: number of friends of $p_i$

bins $b_0, b_1, \ldots, b_{n-1}$; put person $p_i$ in bin $b_{f(i)}$

$\Rightarrow$ $n$ items in $n$ bins: pigeonhole does not apply

• if $b_0$ is empty $\rightarrow$

  $p_1, p_2, \ldots, p_n$ assigned to $b_1, b_2, \ldots, b_{n-1}$

• if $b_0$ is not empty $\rightarrow$ (symmetry) $b_{n-1}$ empty $\rightarrow$

  $p_1, p_2, \ldots, p_n$ assigned to $b_0, b_1, \ldots, b_{n-2}$

$\Rightarrow$ either way there is a "collision"
More general pigeonhole principle

with $N$ items distributed over $k$ bins, there is a bin with at least $\lceil N/k \rceil$ items

select 8 different integers from $\{1,2,\ldots,12\}$, then at least two pairs add up to precisely 13 bins are pairs adding up to 13, thus $k = 6$ bins: $(1,12), (2,11), (3,10), (4,9), (5,8), (6,7)$

items are integers that are selected, thus $N = 8$

⇒ selection corresponds to a choice of bins

⇒ there is a bin with $\lceil N/k \rceil = \lceil 8/6 \rceil = 2$ items

⇒ at least one pair adds up to 13

remove it: $N = 6$, $k = 5$, $\lceil 6/5 \rceil = 2$ ⇒ other pair
Related: cabling, and saving a few cables

connect \( p \) printers to \( d \) desktops (\( d > p \)) such that \( p \) desktops always connect to \( p \) distinct printers, but cheaper than running all \( pd \) cables

printers \( P_1, P_2, \ldots, P_p \), desktops \( D_1, D_2, \ldots, D_d \),

- for \( 1 \leq i \leq p \) connect \( D_i \) to \( P_i : p \) cables
- \( p < k \leq d \) connect \( D_k \) to all \( P_i \)s: \((d-p)p \) cables

\[ \Rightarrow \text{total } (d-p)p+p \] cables (saving \( p^2-p \) cables)

works: \( D_i \) with \( 1 \leq i \leq p \) connects to \( P_i \); if free printers then \( D_k (k > p) \) can connect to them

optimal: with \((d-p)p+p-1 \) cables, there is a \( P_i \) connected to \( \leq d-p \) desktops, thus \( P_i \) not connected to \( \geq p \) desktops: let those print…
Permutations, combinations, etc

in how many different ways can \( r \) objects be selected from collection of \( n \) different objects?

have to distinguish different possibilities:

- may an object be selected more than once?
  \( \Rightarrow \) replacement (repetition) or not (if not: \( r \leq n \))

- is the order of selection relevant?
  \( \Rightarrow \) permutation (“yes”) or combination (“no”)

\( \Rightarrow \) 2×2 different possibilities to be considered:

1. permutation without replacement
2. combination without replacement
3. permutation with replacement
4. combination with replacement
Examples with $n = 10$, $r = 3$

1. permutation without replacement
gold, silver, bronze medal among 10 players
2. combination without replacement
select 3 representatives from class of 10
3. permutation with replacement
select 3-digit PIN
4. combination with replacement
select 3 cookies from 10 types of cookies
or:
number of nonnegative integer solutions
to $x_1 + x_2 + \ldots + x_{10} = 3$ ($x_i \in \mathbb{Z}_{\geq 0}$)
Simple formulas for general $n$ and $r$

1. permutation without replacement: $P(n,r)$
   
   $n$ choices for 1\textsuperscript{st}, $n-1$ for 2\textsuperscript{nd}, \ldots, $n-r+1$ for $r$\textsuperscript{th}
   
   $\Rightarrow P(n,r) = n(n-1)\ldots(n-r+1) = n!/(n-r)!$

2. combination without replacement: $C(n,r)$
   
   each combination can be ordered in $r!$ ways
   
   $\Rightarrow C(n,r)r! = P(n,r) \Rightarrow C(n,r) = \frac{n!}{r!(n-r)!} = \binom{n}{r}$
   
   (pronounced: “$n$ choose $r$”)

3. permutation with replacement
   
   $n$ choices for 1\textsuperscript{st}, $n$ for 2\textsuperscript{nd}, \ldots, $n$ for $r$\textsuperscript{th}
   
   $\Rightarrow$ in total $n^r$ $r$-permutations with repetition

4. combination with replacement
   
   (the only non-intuitive one – do this later)
Examples: combinations without replacement

hands of five cards from standard deck:

52 cards: 13 “kinds” (valeurs) in 4 “suits” (couleurs)
(2,…,10, jack, queen, king, ace; spades, clubs, hearts, diamonds)
(2,…,10, valet, dame, roi, as; pique, trèfle, coeur, carreau)

• how many different hands?
52 choose 5 = \(C(52,5) = \binom{52}{5} = \frac{52!}{5!47!} = 2598960\)

• how many hands contain your favorite card (say 5 of clubs)?
  • pick it, left \(\binom{51}{4} = 249900\) (⇒ almost 10%)
  • or: complement of not picking it:
     \(2598960 - \binom{51}{5} = 249900\)
  • or: \(2598960 \times \left(\frac{5}{52}\right) = 249900\)
More card examples

• # hands containing your two favorite cards?
• pick them, left \( \binom{50}{3} = 19600 \) \( \Rightarrow 0.75\% \)
• or: complement of not picking them:
  \[ 2598960 - \binom{50}{5} = 480200 \]
• not the same, one must be wrong…
• correct version of complement method uses inclusion&exclusion principle:
  \[ 2598960 - \binom{51}{5} - \binom{51}{5} + \binom{50}{5} = 19600 \]
  (subtract card one excluded, subtract card two excluded, add back both excluded)
More card examples

• # hands containing five kinds?
  pick kinds ($C(13,5)$), four suits per kind:
  \[ C(13,5)4^5 = 1317888 \quad (51\%) \]

• # hands with a flush, i.e., all same suit
  pick suit (4), pick 5 out of 13 ($C(13,5)$):
  \[ 4C(13,5) = 5148 \quad (0.2\%) \]

• # hands with four cards of one kind?
  pick kind ($C(13,1)$),
  pick the four cards of that kind ($C(4,4)=1$),
  and pick remaining card ($C(48,1)=48$):
  \[ 13*48 = 624 \quad (0.024\%) \]

• three of a kind: $C(13,1)*4*48*44/2$, (2.11%)
  (or: $C(12,2)*4^2$ instead of $48*44/2$)
Combinatorial and algebraic proofs

- combinatorial proof: formula holds based on counting argument or “insight”
- algebraic proof: usual math manipulations
Combinatorial and algebraic proofs

an $r$-combination from $n$ without replacement is equivalent to
an $(n-r)$-combination from $n$ without replacement

$$\Rightarrow C(n,r) = C(n,n-r)$$

the above is example of a “combinatorial proof”: a counting argument that a formula holds
easily confirmed by a trivial algebraic proof:

$$C(n,r) = \binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)!(n-(n-r))!}$$

$$= \binom{n}{n-r} = C(n,n-r)$$
More examples

1. splitting pile of \( n \) stones:
   - strong induction: total cost \( n(n-1)/2 \)
   - “handshake” argument: same result

3. \( P(n+1,r) = P(n,r)(n+1)/(n+1-r) \)
   - algebraic proof immediate
   - combinatorial: argument:

\[
P(n+1,r+1) = (n+1)P(n,r):
\]
   - take first from \( n+1 \), then \( r \)-perm from \( n \)

or

\[
P(n+1,r+1) = P(n+1,r)(n+1-r):
\]
   - first take \( r \)-perm from \( n+1 \), then take last
More about $n$ choose $r$: binomial coefficients

Pascal’s identity ($0 < k \leq n$): $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$

• combinatorial proof:
pick $k$-combination from $n+1$ by fixing one element: include it ($k-1$ from $n$ remain to be chosen) or don’t ($k$ from $n$ remain to be chosen)

• algebraic proof:
\[
\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!}
\]
\[
= \frac{n!k}{k!(n-k+1)!} + \frac{n!(n-k+1)}{k!(n-k+1)!}
\]
\[
= \frac{n!k + n!(n-k+1)}{k!(n+1-k)!} = \binom{n+1}{k}
\]
**Binomial theorem**

for $n \geq 0$:

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

- combinatorial proof:
  expand product $(x+y)^n$: for the term $x^{n-k}y^k$
  the $y$ needs to be chosen $k$ out of $n$ times
  (order irrelevant since $x^{n-k}y^k = yx^{n-k}y^{k-1} = \ldots = y^kx^{n-k}$)
  $\Rightarrow$ coefficient of $x^{n-k}y^k$ must be $n$ choose $k$

- algebraic proof:
  use mathematical induction
  and Pascal’s identity
Algebraic proof of binomial theorem

let $P(n)$ be the assumption that \[(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k\]

- $(x + y)^0 = 1 = \binom{0}{0} x^0 y^0 = \sum_{k=0}^{0} \binom{0}{k} x^{0-k} y^k$
  shows that $P(0)$ holds
- assume $P(n)$ holds for some $n \geq 0$. Then

$$(x + y)^{n+1} = (x + y) \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \text{ (used induction hypothesis)}$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{n-k+1} y^k + \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^{k+1}$$

$$= \binom{n}{0} x^{n+1} + \binom{n}{1} x^{n-1} y^1 + \sum_{k=1}^{n-1} \binom{n}{k} x^{n-k} y^{k+1} + \binom{n}{n} y^{n+1}$$

$$= \binom{n+1}{0} x^{n+1} + \sum_{k=1}^{n} \binom{n}{k} x^{n+1-k} y^k + \sum_{k=1}^{n} \binom{n}{k-1} x^{n+1-k} y^k + \binom{n+1}{n+1} y^{n+1}$$

$$= \binom{n}{0} x^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} x^{n+1-k} y^k + \binom{n+1}{n+1} y^{n+1} \text{ (Pascal's identity)}$$

$$= \sum_{k=0}^{n+1} \binom{n+1}{k} x^{n+1-k} y^k$$
Combinatorial and algebraic proofs

- combinatorial proof: formula holds based on counting argument or “insight”
- algebraic proof: usual math manipulations

seen both types of proofs for
- Pascal’ identity $(0 < k \leq n)$:
  \[
  \binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}
  \]
- binomial theorem: for $n \geq 0$
  \[
  (x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k
  \]
Consequence of binomial theorem

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

- algebraic proofs:
  - take $x = y = 1$ in binomial theorem
  - mathematical induction, Pascal’s identity

- combinatorial proof:
  - $2^n$ is the number of length $n$ bitstrings
  - write the number of length $n$ bitstrings as $\sum_{k=0}^{n} C_i$, where $C_i$ is the number of length $n$ bitstrings with $i$ bits “on”, and note that $C_i$ is $n$ choose $I$ (or look at subsets and their cardinalities)
Another consequence of binomial theorem

\[ \binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k} \binom{n}{k} \text{ for } r < m, n \]

(Vandermonde's identity)

- **algebraic proof:**
  use \((x+y)^{m+n} = (x+y)^m(x+y)^n\)
  with binomial theorem
  and compare the terms for \(x^{m+n-r}y^r\)

- **combinatorial proof:**
  count cardinality \(r\) subsets of cardinality \(m+n\) set in different ways

- **consequence:** \(\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2\)
Final combinatorial ↔ algebraic example

\[(\binom{n}{r})(\binom{r}{k}) = (\binom{n}{k})(\binom{n-k}{r-k}), \quad 0 \leq k \leq r \leq n\]

- combinatorial proof: suppose you need to pick a committee of \(r\) out of \(n\), and a subcommittee of \(k\) out of those \(r\) (LHS). or pick the subcommittee of \(k\) first, then remaining \(r-k\) from remaining \(n-k\) (RHS)

- algebraic proof straightforward too:

\[
\binom{n}{r}\binom{r}{k} = \frac{n!}{r!(n-r)!} \cdot \frac{r!}{k!(r-k)!} = \frac{n!}{(n-r)!k!(r-k)!} = \frac{(n-r)!k!(r-k)!}{k!(n-k)! (r-k)!(n-r)!} = \binom{n}{k}\binom{n-k}{r-k}
\]
Useful identity (for combination with repetition)
\[ \binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r} \quad \text{for} \quad r \leq n \]

- Combinatorial proof: look at last “on” bit of \( r+1 \) “on” bits in \( n+1 \) positions
More precisely

combinatorial proof of \( \binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r} \ (r \leq n) \)

pick \( r+1 \) out of \( x_1, x_2, \ldots, x_{n+1} \),

- largest index is \( n+1 \)
  
  or \( \binom{n}{r} \) ways to pick other \( r \)

- largest index is \( n \)
  
  or \( \binom{n-1}{r} \) ways to pick other \( r \)

- largest index is \( n-1 \)
  
  or \( \binom{n-2}{r} \) ways to pick other \( r \)

- largest index is \( n-2 \)
  
  or \( \binom{n-3}{r} \) ways to pick other \( r \)

\[ \vdots \]

- largest index is \( r+1 \)
  
  or \( \binom{r}{r} \) ways to pick other \( r \)

All possibilities disjoint
Useful identity

\[(n+1)_{r+1} = \sum_{j=r}^{n} \binom{j}{r} \quad \text{for} \quad r \leq n\]

- combinatorial proof: look at last “on” bit of \(r+1\) “on” bits in \(n+1\) positions
- proof by hand waving: repeatedly use Pascal’s identity, walking up Pascal’s triangle
- algebraic proof: use mathematical induction with respect to \(n\) (formalizing hand waving)
  - for \(n = r\) identity holds
  - assume holds for \(n\); then for \(n+1\):
    \[(n+2)_{r+1} = (n+1)_r + (n+1)_{r+1} \quad \text{(use Pascal's identity)}\]
    \[= (n+1)_r + \sum_{j=r}^{n} \binom{j}{r} = \sum_{j=r}^{n+1} \binom{j}{r}\]
Back to counting formulas: pick $r$ from $n$

1. Permutation without replacement: $P(n,r)$
   \[ P(n,r) = n(n-1)\ldots(n-r+1) = \frac{n!}{(n-r)!} \]

2. Combination without replacement: $C(n,r)$
   \[ C(n,r) = \frac{n!}{r!(n-r)!} = \binom{n}{r} \]

3. Permutation with replacement
   \[ n^r \]

4. Combination with replacement
   still not done, and a bit less intuitive
Pick $r$-combination from $n$ with replacement to develop intuition, a few basic examples:

- given infinite supply of $n = 1$ cookie type, in how many ways can one pick $r$ cookies? clearly only one way: $r$ cookies of type 1
  
  \[ n = 1: \text{constant in } r \]

- same question with $n = 2$ types of cookies: from 0 to $r$ of type 1, others type 2: $r+1$ ways
  \[ n = 2: \text{linear in } r \]

- same question with $n = 3$ types of cookies: $s, 0 \leq s \leq r$, of type 1: $r-s$ of types 2 or 3, thus
  \[ \sum_{s=0}^{r} (r - s + 1) = (r + 1)(r + 2) / 2 \] ways
  \[ n = 3: \text{quadratic in } r \]
$r$-combination from $n$ with replacement

observations we made:

- $n = 1$: 1 way
- $n = 2$: $r+1$ ways
- $n = 3$: $(r+1)(r+2)/2$ ways

⇒ suggests $(n+r−1)$ choose $n−1$ ways (✳)
⇒ let $f(n,r)$ denote the number of $r$-combinations from $n$ with replacement, then

$$f(n,r) = f(n−1,0)+f(n−1,1)+\ldots+f(n−1,r):$$

take $r$ of type 1, 0 left to take of $n−1$ types

take $r−1$ of type 1, 1 left to take of $n−1$ types

take $r−2$ of type 1, 2 left to take of $n−1$ types

... take 0 of type 1, $r$ left to take of $n−1$ types
\textbf{r-combination from }n\textbf{ with replacement observations we made:}

- \( n = 1 \): 1 way
  
  \( n = 2 \): \( r+1 \) ways
  
  \( n = 3 \): \((r+1)(r+2)/2\) ways

\( \Rightarrow \) suggests \((n+r-1)\) choose \(n-1\) ways (\(\ast\))

\( \Rightarrow \) let \( f(n,r) \) denote the number of \(r\)-combinations from \(n\) with replacement, then

\[
f(n,r) = f(n-1,0) + f(n-1,1) + \ldots + f(n-1,r)
\]

- induction proof of \(\ast\): basis \(n = 1\) is okay;

\[
f(n,r) = \sum_{s=0}^{r} f(n-1,s) = \sum_{s=0}^{r} \binom{n+s-2}{n-2}
\]

\((\text{use } n + s - 2 = j) = \sum_{j=n-2}^{n+r-2} \binom{j}{n-2} = \binom{n+r-1}{n-1}\)
$r$-combination from $n$ with replacement
why is the result so simple?
combinatorial proof that $f(n, r) = \binom{n+r-1}{n-1}$
uses $n+r-1$ positions $n-1$ of which are separators that “switch” to next type
note:

- $f(n, r)$ counts number of nonnegative integer solutions to $x_1 + x_2 + \ldots + x_n = r$ ($x_i \in \mathbb{Z}_{\geq 0}$)
little tricks to deal with:
  - $x_i \geq b_i$ for bounds $b_i$: use $r - \sum_{i=1}^{n} b_i$
  - $x_1 + x_2 + \ldots + x_n \leq r$: use slack variable $x_{n+1}$
  - $C(n, r)$ product for indistinguishable objects