Problem Set 13

Date: 12.12.2014 Not graded

Problem 1. You are playing the following game of chance. In each turn you place a bet of x francs, after which a fair coin is flipped. If it produces *tails*, you receive your bet x francs plus an extra x francs; otherwise you lose the x francs from the bet. You have the following strategy: in the first turn you bet 1 francs; then every time you lose, you double your bet. If you cannot afford to double your bet, you leave the game. In case you win, you also stop the game and walk away with the money.

(a) If you have at disposal 255 francs, what is the probability that you walk away broke? What is the expected value of your total gains? (In case of losses, gains are negative)

(b) Do the same for the case where you have an infinite supply of money.

Problem 2.

(a) During an oral exam, an Evil Teaching Assistant (ETA) prepares 2 very hard questions, and 1 very easy question and puts them in 3 identical boxes. The student is then allowed to pick one box, without looking inside. In order to maximize his chances of getting a perfect grade, the student would like to pick the box containing the very easy question. What is the probability of picking the very easy question?

(b) The ETA announces to the student that he will remove one of the hard questions. He opens one of the two remaining boxes, and proceeds to read the question to the student, confirming that it was indeed a very hard question. The ETA then offers the student the choice to exchange the box he picked at the beginning for the remaining, unpicked box. Would you keep the original box or exchange it? What is the probability that the very easy question is in the unpicked box?

Problem 3.

(a) Suppose X encodes the outcome of a fair coin toss in the following way: if *tail* is produced $X = 1$ and if *head* is produced $X = 0$. In addition Y encodes the outcome of a biased coin tossed *independent of the first coin* using the same convention. (i.e. $Y = 1$ means the outcome of the coin toss was *tail* while $Y = 0$ means the outcome was *head*). Let

$$Z = X + Y \pmod 2$$

and note that Z also takes values in $\{0, 1\}$. Show that Z takes values $\{0, 1\}$ with equal probability no matter how biased the second coin is.

(b) Now let's generalize part (a). Suppose X is the outcome of a q-sided equiprobable die whose faces are labeled from 0 to $q - 1$. In other words, $X \in \{0, 1, \ldots, q - 1\}$ and takes each of those values with equal probability of $\frac{1}{q}$. Furthermore Y is the outcome of another q-sided dice which we don't know whether it is equiprobable or not. Namely, Y takes values in $\{0, 1, \ldots, q - 1\}$ but we don't know its probability distribution. Let

$$Z = X + Y \pmod q$$

Show that Z takes values in $\{0, 1, \ldots, q - 1\}$ with equal probability regardless of the distribution of Y.

(c) Suppose in part (b) we redefine Z as

$$Z = X \cdot Y \pmod{q},$$

where q is a prime number. Does Z still have uniform distribution in $\{0, 1, \ldots, q-1\}$?

Problem 4. Consider the setting in Problem 3 (b).

(a) Show that for any $b, c \in \{0, 1, \ldots, q-1\}$ the events $\{Y = b\}$ and $\{Z = c\}$ are independent.

(b) Suppose further that Y is also the outcome of an equiprobable die. Show that for any $a \in \{0, 1, \ldots, q-1\}$ the events $\{X = a\}$ and $\{Z = c\}$ are also independent.

(c) Are the events $\{X = a\} \cap \{Y = b\}$ and $\{Z = c\}$ independent for any $a, b, c \in \{0, 1, \ldots, q-1\}$?

Problem 5. Urn 1 contains 2 blue tokens and 8 red tokens; urn 2 contains 12 blue tokens and 3 red tokens. You roll a die to determine which urn to choose: if you roll a 1 or 2 you choose urn 1; if you roll a 3, 4, 5, or 6 you choose urn 2. Once the urn is chosen, you draw out a token at random from that urn. Given that the token is blue, what is the probability that the token came from urn 1?

Problem 6. This year, 17% of the population of Lausanne vaccinated against the flu. It has been calculated that the people that are not vaccinated get the flu with probability 0.12, and the people that are vaccinated with probability 0.02.

(a) What is the probability of getting the flu?

(b) What is the probability that a person that has the flu was vaccinated?