Problem Set 12

Date: 05.12.2014 Not graded

Problem 1.

(a) What is the number of divisors of 12?

(b) What is the number of divisors of 36?

(c) Let $m \in \mathbb{N}$ and assume that the unique prime factorization of m is

$$m = \prod_{i=1}^{k} p_i^{\alpha_i},$$

for some prime numbers $p_1, \cdots, p_k$ and some positive integers $\alpha_1, \cdots, \alpha_k$. What is the number of divisors of m?

Problem 2. A vending machine dispensing books of stamps accepts only \$1 coins, \$1 bills and \$2 bills. Let a_n denote the number of ways of depositing n dollars in the vending machine, where the *order* in which the coins and bills are deposited *matters.*

(a) Find a recurrence relation for a_n and give the necessary initial condition(s).

(b) Find an explicit formula for a_n by solving the recurrence relation in part (a).

(c) Now, let b_n denote the number of ways of depositing n dollars in the vending machine, assuming that the *order* in which the coins and bills are deposited *does not matter.* For example, you can pay 2\$ in 4 ways: two 1\$ coins; two 1\$ bills; one 1\$ coin and one 1\$ bill; one 2\$ bill. Find the value of b_n.

Problem 3.

(a) Let a_n be a sequence such that $a_n = \Theta\left(\left(\frac{1}{2}\right)^n\right)$. Let $b_n = \sum_{i=1}^{n} a_i$. Find a Θ-approximation for the sequence b_n.

(b) Assume now that $a_n = \Theta\left(2^n\right)$ and define, again, $b_n = \sum_{i=1}^{n} a_i$. Find a Θ-approximation for the sequence b_n.

Hint. Generating functions ...

Problem 4. Assume that the generating function associated to the sequence a_n is $F(x)$. Find the generating function associated to b_n, where b_n is defined as below:

(a) $b_n = 2a_n - a_{n+1}$,

(b) $b_n = n \cdot a_n$,

(c) $b_0 = a_0$ and $b_n = \dfrac{a_n}{n}$ for $n \geq 1$.

Problem 5. You throw a fair coin $2n$ times, where $n \in \mathbb{N}_{\geq 1}$.

(a) What is the probability of getting $2n$ Heads?

(b) What is the probability of getting 2 Tails?

(c) Suppose that you want to bet with a friend on the number of Tails that you are going to observe. On what number should you bet?

(d) Suppose now that the coin is biased such that the probability of getting Tails in a single trial is $\frac{1}{3}$. The $2n$ trials remain independent and equiprobable. Compute again the probabilities of point (a) and (b).

Problem 6. If we sum the outcomes of two dice with 6 equiprobable faces, we obtain the integers from 2 to 12 with different probabilities. The aim of this problem is to show that it is impossible to modify the probabilities of obtaining each face of each die so that the sum of the outcomes of the dice is a uniform random variable with values in $\{2, 3, \cdots, 12\}$.

Suppose by contradiction that the claim is false. Let p_i and q_i be the probabilities that the face i is obtained in the first and the second die, respectively ($i \in \{1, 2, \cdots, 6\}$).

(a) As a function of p_i and q_i ($i \in \{1, 2, \cdots, 6\}$), what is the probability that the sum of the outcomes is 2? Since the sum of the outcomes of the dice is uniform in $\{2, 3, \cdots, 12\}$, what condition do we obtain on the p_i and the q_i?

(b) Same question for the sum of the outcomes equal to 12.

(c) Let a, $b \in \mathbb{R}_{\geq 0}$. Prove that

$$\frac{a+b}{2} \geq \sqrt{ab}.$$

(d) What is the probability that the sum of the outcomes of the two dice is 7? Bound this result with a function of p_1, q_1, p_6, and q_6. Then, using the previous points, find a contradiction and prove the claim.

(e) The aim of this part is to show an alternative proof of the claim which uses generating functions. Let $p(x) = \sum_{i=1}^{6} p_i x^i$ and $q(x) = \sum_{i=1}^{6} q_i x^i$ be the generating functions associated to the probability distributions of the first and the second die, respectively.

What is the generating function $s(x)$ of the sum of the two dice? How can you write it using the fact that the sum of the outcomes is a uniform random variable? From these consideration, deduce a contradiction.

Hint 1. How many real roots does the polynomial $\sum_{i=0}^{10} x^i$ have?

Hint 2. Recall that a polynomial of odd degree has always at least 1 real root.