Discrete Structures EPFL, Fall 2014

Problem Set 5

Date: 17.10.2014 Not graded

Problem 1. Use the definition of big-O to prove that:

a) $\pi n^4 + 10n^3 + 10^{10}n^2 + 10^{10^{10}}n$ is $O(n^4)$.

b) $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + (n-1) \cdot n$ is $O(n^3)$.

c) $1^{2014} + 2^{2014} + 3^{2014} + \cdots + n^{2014}$ is $O(n^{2015})$.

d) $\lceil n + \sqrt{7} \rceil \cdot \lfloor n^2 - \sqrt{2} \rfloor + \lceil n^4 + \sqrt{3} \rceil$ is $O(n^4)$.

Problem 2. Suppose you have two different algorithms for solving a problem. To solve a problem of size n, the first algorithm uses exactly $n\sqrt{n}$ operations and the second algorithm uses exactly $n^2 \log n$ operations. As n grows, which algorithm uses fewer operations?

Problem 3. Consider the following algorithm:

Algorithm 1

Require: n: positive integer

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1: t ← 0
2: for i = 1 to n^2 do
3:     for j = 1 to i do
4:         t ← (it + jt + 1)^2
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How many additions are performed as a function of n? First, write the exact formula and, then, provide a Θ approximation.

Problem 4. For each of the following tasks, find a function $f(n)$ such that the required number of operations is $\Theta(f(n))$.

a) Compute the sum of the elements of a vector of length n.

b) Multiply an $n \times n$ matrix with a vector of length n.

c) Multiply an $n \times n$ matrix with an $n \times n$ matrix.

d) Multiply an $n \times \lfloor \sqrt{n} \rfloor$ matrix with an $\lfloor \sqrt{n} \rfloor \times n$ matrix.

Problem 5. Write whether each of the following statements is **True** or **False**:

a) $\frac{1}{n^2}$ is $\Omega\left(\frac{1}{n^2}\right)$.

b) For any $a, b \in \mathbb{R}$ s.t. $a < b$, $\frac{1}{n^a}$ is $o\left(\frac{1}{n^b}\right)$.

c) $\log(1+n)$ is $O(\sqrt{n})$.

d) $n^4 4^n$ is $o\left(\frac{1}{n^5} 5^n\right)$.

e) 2^{n^2} is $O\left(4^n 3^{n \log n}\right)$.

f) $\frac{n!}{2^n}$ is $\Omega(2^n)$.

g) $(n!!)^2$ is $\Omega(n!)$, where $n!!$ denotes the *semifactorial* of n defined as follows: if n is even, $n!!$ is the product of all the strictly positive even integers smaller or equal than n. if n is odd, then $n!!$ is the product of all strictly positive odd integers smaller or equal than n.

Problem 6. Find all pairs of functions in the following list that are of the same order:
$n^2 + \log n,\ 2^n + 3^n,\ 100n^3 + n^2,\ n^2 + 2^n,\ n^2 + n^3,\ 3n^3 + 2^n$.