Discrete Structures EPFL, Fall 2014

Solution to Problem Set 7

Date: 31.10.2014 Not graded

Problem 1.

a) The multiplicative inverse of 7 modulo 11 exists (and is equal to 8). The multiplicative inverse of 6 modulo 8 doesn't exist. The multiplicative inverse of 5 modulo 8 exists (and is equal to 5).

b) Let x be the multiplicative inverse of a modulo b. That is, $ax \equiv 1 \pmod{b}$ or equivalently,

$$ax = bk + 1 \qquad \text{for some integer } k$$

which is equivalent to

$$ax - bk = 1.$$

Take $y := -k$ and recall that Bézout Lemma states that *every integer of the form $ax + by$ is a multiple of the greatest common divisor of a and b, $d := \gcd(a, b)$.* Consequently, we can find an integer x such that $ax \equiv 1 \pmod{b}$ *if and only if* $\gcd(a, b) = 1$.

Swapping the roles of a and b we can conclude that $\gcd(a, b) = 1$ is also a necessary and sufficient condition for existence of the multiplicative inverse of b modulo a.

For the previous examples we can check that

- $\gcd(7, 11) = 1$ hence the multiplicative inverse of 7 modulo 11 exists.
- $\gcd(6, 8) = 2 \neq 1$ hence the multiplicative inverse of 6 modulo 8 doesn't exist.
- $\gcd(5, 8) = 1$ hence the multiplicative inverse of 5 modulo 8 exists.

c) Recall the Euclid Algorithm to find the greatest common divisor of two numbers a and b. At each step $k = 0, 1, \ldots,$ the algorithm finds the quotient q_k and remainder r_k such that

$$r_{k-2} = q_k r_{k-1} + r_k,$$

starting with $r_{-2} := a$ and $r_{-1} := b$. In other words, the algorithm produces a sequence of quotients and reminders as:

$$\begin{aligned}
a &= q_0 b + r_0 && \text{(at step } k = 0)\\
b &= q_1 r_0 + r_1 && \text{(at step } k = 1)\\
r_0 &= q_2 r_1 + r_2 && \text{(at step } k = 2)\\
r_1 &= q_3 r_2 + r_3 && \text{(at step } k = 3)\\
&\vdots
\end{aligned}$$

and terminates at some step N when $r_N = 0$. The last non-zero remainder is $d := \gcd(a, b)$. That is,

$$\begin{aligned}
r_{N-3} &= q_{N-1} r_{N-2} + d && \text{(at step } k = N - 1)\\
r_{N-2} &= q_N d + 0 && \text{(at step } k = N)
\end{aligned}$$

Rewriting the equation of step $N-1$, we have

$$d = r_{N-3} - q_{N-1} r_{N-2}.$$

Now, we can use the equation for step $N-2$ to write $r_{N-2} = r_{N-4} - q_{N-2} r_{N-3}$ and replace this in the above equation to get:

$$d = (1 + q_{N-1} q_{N-2}) r_{N-3} - q_{N-1} r_{N-4}$$

We can again use the equation for step $N-3$ and write $r_{N-3} = r_{N-5} - q_{N-3} r_{N-4}$ and replace r_{N-3} in the above equation to get:

$$d = (1 + q_{N-1} q_{N-2}) r_{N-5} - (q_{N-3} + q_{N-1} q_{N-2} q_{N-3} + q_{N-1}) r_{N-4}$$

Continuing this procedure up to very first step $k = 0$, we will be able to write d as a linear combination of $r_{-2} = a$ and $r_{-1} = b$:

$$d = sa - tb$$

Now, if $d = \gcd(a, b) = 1$, we have found numbers s and t such that

$$sa = 1 + bt$$

which means s is the multiplicative inverse of a modulo b: $sa \equiv 1 \pmod{b}$.

d) i. Running the Euclid algorithm on the pair of integers 148 and 57 we have

$$\begin{aligned} 148 &= 2 \times 57 + 34, \\ 57 &= 1 \times 34 + 23, \\ 34 &= 1 \times 23 + 11, \\ 23 &= 2 \times 11 + 1, \end{aligned}$$

(note that we have not written down the very last trivial step). Hence, starting from the last equation and going back to top, we will have

$$\begin{aligned} 1 &= 23 - 2 \times 11 \\ &= 23 - 2 \times (34 - 1 \times 23) \\ &= 3 \times 23 - 2 \times 34 \\ &= 3 \times (57 - 1 \times 34) - 2 \times 34 \\ &= 3 \times 57 - 5 \times 34 \\ &= 3 \times 57 - 5 \times (148 - 2 \times 57) \\ &= 13 \times 57 - 5 \times 148 \end{aligned}$$

which shows $13 \times 57 \equiv 1 \pmod{148}$.

ii. Running the Euclid algorithm on the pair of integers 341 and 123 we have

$$\begin{aligned} 341 &= 2 \times 123 + 95, \\ 123 &= 1 \times 95 + 28, \\ 95 &= 3 \times 28 + 11, \\ 28 &= 2 \times 11 + 6, \\ 11 &= 1 \times 6 + 5, \\ 6 &= 1 \times 5 + 1. \end{aligned}$$

Thus,

$$\begin{aligned}
1 &= 6 - 1 \times 5 \\
&= 6 - 1 \times (11 - 1 \times 6) \\
&= 2 \times 6 - 1 \times 11 \\
&= 2 \times (28 - 2 \times 11) - 1 \times 11 \\
&= 2 \times 28 - 5 \times 11 \\
&= 2 \times 28 - 5 \times (95 - 3 \times 28) \\
&= 17 \times 28 - 5 \times 95 \\
&= 17 \times (123 - 95) - 5 \times 95 \\
&= 17 \times 123 - 22 \times 95 \\
&= 17 \times 123 - 22 \times (341 - 2 \times 123) \\
&= 61 \times 123 - 22 \times 341
\end{aligned}$$

which shows $61 \times 123 \equiv 1 \pmod{341}$.

iii. Running the Euclid algorithm on the pair of integers 921 and 257 we have

$$\begin{aligned}
921 &= 3 \times 257 + 150, \\
257 &= 1 \times 150 + 107, \\
150 &= 1 \times 107 + 43, \\
107 &= 2 \times 43 + 21, \\
43 &= 2 \times 21 + 1.
\end{aligned}$$

Therefore,

$$\begin{aligned}
1 &= 43 - 2 \times 21 \\
&= 43 - 2 \times (107 - 2 \times 43) \\
&= 5 \times 43 - 2 \times 107 \\
&= 5 \times (150 - 1 \times 107) - 2 \times 107 \\
&= 5 \times 150 - 7 \times 107 \\
&= 5 \times 150 - 7 \times (257 - 1 \times 150) \\
&= 12 \times 150 - 7 \times 257 \\
&= 12 \times (921 - 3 \times 257) - 7 \times 257 \\
&= 12 \times 921 - 43 \times 257
\end{aligned}$$

Hence, $-43 \times 257 \equiv 1 \pmod{921}$ which means the multiplicative inverse of 257 is $-43 \equiv 878 \pmod{921}$.

Problem 2.

a) The cardinality of $\mathcal{A}_n$ and the number of terms on the left hand side of (1) is $(n+1)^n$. By the uniqueness of the factorization, for each element m of $\mathcal{A}_n$, the term $1/m$ appears in the expansion of the product on the left. Thus, the expansion of this product is a rearrangement of the finite sum on the right.

b) Recall that for any $a \neq 1$,

$$\sum_{i=0}^{n} a^i = \frac{1 - a^{n+1}}{1 - a}.$$

Hence,

$$1+\frac{1}{p_j}+\ldots+\frac{1}{p_j^n}=\sum_{i=0}^{n}\frac{1}{p_j^i}=\frac{1-\frac{1}{p_j^{n+1}}}{1-\frac{1}{p_j}}\leq\frac{1}{1-\frac{1}{p_j}}=\frac{p_j}{p_j-1}=1+\frac{1}{p_j-1}$$

c) Using (1), we obtain that $\ln\sum_{m\in\mathcal{A}_n}\frac{1}{m}=\ln\prod_{i=1}^{n}\sum_{j=0}^{n}\frac{1}{p_i^j}$. Since $\ln(\cdot)$ is a monotonous function, using point b), we have that

$$\ln\prod_{i=1}^{n}\sum_{j=0}^{n}\frac{1}{p_j^n}\leq\ln\prod_{i=1}^{n}\left(1+\frac{1}{p_i-1}\right)=\sum_{i=1}^{n}\ln\left(1+\frac{1}{p_i-1}\right).$$

We can check that for $x\geq 0$, $\ln(1+x)\leq x$.[1] As a result,

$$\sum_{i=1}^{n}\ln\left(1+\frac{1}{p_i-1}\right)\leq\sum_{i=1}^{n}\frac{1}{p_i-1}$$

Putting all these results together, we obtain $\displaystyle\sum_{i=1}^{n}\frac{1}{p_i-1}\geq\ln\sum_{m\in\mathcal{A}_n}\frac{1}{m}$.

d) $\{1,\ldots,n\}\subseteq\mathcal{A}_n$ because for all $j\in\{1,\ldots,n\}$, the unique factorization of j contains only primes from $\{p_1,\ldots,p_n\}$ as $p_n\geq n$. Also the multiplicity of each prime needs to be at most n, since $p_i^{n+1}\geq 2^{n+1}>n$.

This proves that

$$\ln\sum_{m\in\mathcal{A}_n}\frac{1}{m}\geq\ln\sum_{m=1}^{n}\frac{1}{m}.$$

As concerns the left hand side, note that for $j\geq 2$,

$$\frac{1}{p_j-1}\leq\frac{1}{p_{j-1}}.$$

In addition, if $j=1$, then

$$\frac{1}{p_1-1}=\frac{1}{2-1}=1.$$

Hence $\displaystyle\sum_{j=1}^{n}\frac{1}{p_j-1}\leq 1+\sum_{j=2}^{n}\frac{1}{p_{j-1}}=1+\sum_{j=1}^{n-1}\frac{1}{p_j}$.

e) We already know that $\displaystyle\sum_{j=1}^{n}\frac{1}{j}\geq\ln(n)\geq\ln(n-1)$ Hence using the upper-bound of (2),

$$\sum_{j=1}^{n-1}\frac{1}{p_j}\geq\ln(\ln(n-1))-1$$

which shows $\sum_{j=1}^{n}\frac{1}{p_j}=\Omega(\log\log n)$.

Problem 3.

[1]Define $f(x)=\ln(x+1)$ and $g(x)=x$. Then $g(0)=f(0)=0$ and $g'(x)=1>\dfrac{1}{1+x}=f'(x)$ for any $x\geq 0$. Consequently $f(x)\leq g(x)$ for any $x\geq 0$.

a) **Base Step:** The claim clearly holds for $n = 1$, $2^2 - 1 = 3 \mid 3$.

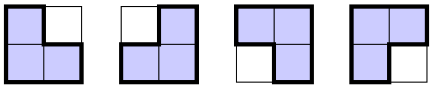
Induction Step: Assume $2^{2n} - 1 \mid 3$. That is $2^{2n} = 3k + 1$ for some integer k. Then $2^{2(n+1)} - 1 = 2^{2n} \times 4 - 1 = 12k + 4 - 1 = 12k + 3 \mid 3$. $\square$

b) **Base Step:** The claim clearly holds for $n = 1$, $(a - b) \mid (a - b)$.

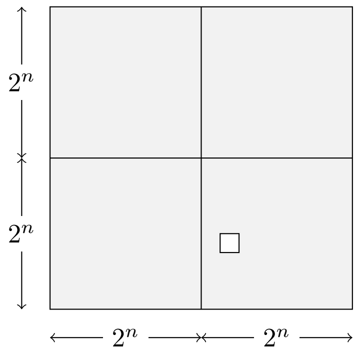
Induction Step: Assume $(a^n - b^n) \mid (a - b)$. We can always write $a^{n+1} - b^{n+1} = a^{n+1} - a^n b + a^n b - b^n = a^n(a - b) + (a^n - b^n)b \mid (a - b)$ because of the assumption $(a^n - b^n) \mid (a - b)$ (and also the base case $(a - b) \mid (a - b)$). $\square$

Problem 4.

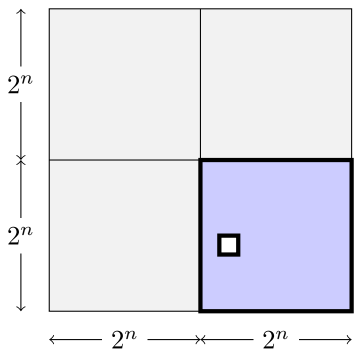
Base Step: Any shape in $\mathcal{C}_1$ is clearly *cool*. No matter which square is missing, any shape in $\mathcal{C}_1$ can be tiled using a single L-shaped tile:



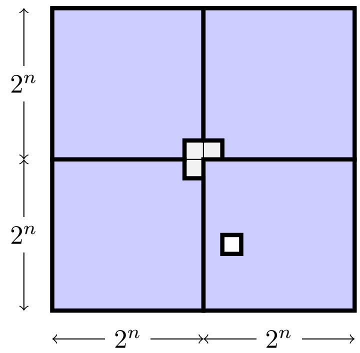
Induction Step: Assume that any shape in $\mathcal{C}_n$ is *cool*. In other words, we can tile a $2^n \times 2^n$ grid using L-shaped tiles leaving one empty 1×1 square no matter which square is missing. We shall show that any shape in $\mathcal{C}_{n+1}$ is *cool*, namely, that we can tile a $2^{n+1} \times 2^{n+1}$ grid using L-shaped tiles leaving one empty 1×1 square no matter which square is missing. The $2^{n+1} \times 2^{n+1}$ grid consists of four $2^n \times 2^n$ grids placed side by side. The square that we want to be empty is hence in one of these four $2^n \times 2^n$ sub-grids. Assume without loss of generality this is the bottom right grid:



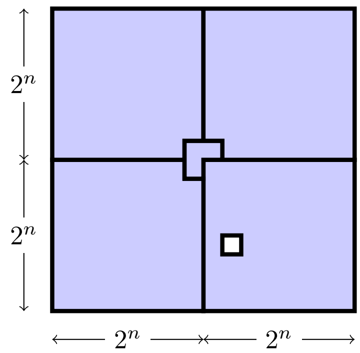
By the induction assumption, we can tile this sub-grid leaving the desired square empty:



Furthermore, again using the induction assumption, we can tile each of the three remaining $2^n \times 2^n$ grids using L-shaped tiles leaving the closest square to the center empty:



This leaves us with a single L-shaped untiled area in the center which can be tiled using an additional L-shaped tile:



Hence, any shape in $\mathcal{C}_{n+1}$ is *cool*. $\square$

Problem 5. Suppose $n = 35$ and we are proving the claim for $n + 1 = 36$. 36 is not prime but $36 = 3 \times 12$. By the induction hypothesis 12 has a prime factorization $12 = p_1p_2p_3$ and 3 is prime hence $36 = 3p_1p_2p_3$. However, $36 = 4 \times 9$ as well and by the induction hypothesis we again have $4 = q_1q_2$ and $9 = r_1r_2$, thus $36 = q_1q_2r_1r_2$ as well. The question is how we know that $3, p_1, p_2$, and p_3 are the same prime numbers as q_1, q_2, r_1, and r_2 (up to a permutation)? They indeed are, but this does not follow from the induction hypothesis. This is called a *breakdown error*. If we try to show that something is unique and we break it down (as we broke down $n + 1 = rs$) we need to argue that nothing changes if we break it down a different way (i.e. $n + 1 = tu$).