Problem 1.

a) \( \pi n^4 + 10n^3 + 10^{10}n^2 + 10^{10}n = (\pi + \frac{10}{n} + \frac{10^{10}}{n} + \frac{10^{10}}{n})n^4 \leq 7n^4 \) for \( n \geq k := \lceil \sqrt{\sqrt{10^{10}}} \rceil \) (since for such \( n \)’s all the three fractions inside the parenthesis are less than 1).

b) \( 1 \cdot 2 + 2 \cdot 3 + \cdots + (n - 1) \cdot n \leq (n - 1) \cdot n + (n - 1) \cdot n + \cdots + (n - 1) \cdot n = (n - 1)^2 \cdot n \leq n^3. \)

c) \( \sum_{i=1}^{n} \frac{1}{2} \cdot n^2 = n^2(n^2 + 1) = n^2(n^2 + 1) = \sum_{i=1}^{n} \frac{1}{2} \cdot n^2. \)

d) Using the identities \( \lfloor x \rfloor \leq x \) and \( \lceil x \rceil \leq x + 1 \) we can write,

\[
\left\lfloor \frac{n + \sqrt{7}}{2} \right\rfloor \cdot \left\lfloor \frac{n^2 - \sqrt{2}}{2} \right\rfloor + \left\lfloor \frac{n^3 + \sqrt{3}}{2} \right\rfloor \leq (n + \sqrt{7} + 1)(n^2 - \sqrt{2}) + (n^3 + \sqrt{3})
\]
\[
= n^4 + n^3 + (\sqrt{7} + 1)n^2 - \sqrt{2}n + (\sqrt{3} - \sqrt{2} - \sqrt{7}\sqrt{2})
\]
\[
= \left(1 + \frac{1}{n} + \frac{\sqrt{7} + 1}{n^2} - \frac{\sqrt{2}}{n^3} + \frac{\sqrt{3} - \sqrt{2} - \sqrt{7}\sqrt{2}}{n^4}\right) \cdot n^4
\]
\[
\leq 11n^4, \quad \text{for } n \geq 1.
\]

Problem 2. It is clear that the first algorithm uses fewer operations as \( n \) grows. In fact it can easily checked that \( \lim_{n \to +\infty} \frac{n^{\sqrt{n}}}{n^2 \log n} = \lim_{n \to +\infty} \frac{1}{\sqrt{n} \log n} = 0. \)

Problem 3. Each execution of line 4 involves 2 additions. Consequently each round of the inner for loop requires \( 2i \) additions. Thus, the total number of additions performed in the algorithm is

\[
\sum_{i=1}^{n^2} 2i = 2 \times \frac{1}{2} n^2(n^2 + 1) = n^2(n^2 + 1).
\]

Furthermore \( n^2(n^2 + 1) = \Theta(n^4) \) because,

\[
n^4 \leq n^2(n^2 + 1) \leq 2n^4
\]

for all \( n \geq 1. \)

Problem 4.

a) We need \( n - 1 \) additions to compute the sum of the elements of a length \( n \) vectors and \( n - 1 = \Theta(n). \)

b) For multiplying each row by the vector, we need \( n \) multiplications and \( n - 1 \) additions. Thus, in total we need \( n^2 \) multiplications and \( n(n - 1) \) additions which means in total \( 2n^2 - n \) operations. \( 2n^2 - n = \Theta(n^2). \)

c) We need to repeat the task of b) \( n \) times. Hence we need \( n^3 - n^2 = \Theta(n^3) \) operations in total.
d) The result is a $n \times n$ matrix. To compute each element of the result, we need $\lceil \sqrt{n} \rceil$ multiplications and $\lceil \sqrt{n} \rceil - 1$ additions. Thus, in total, we need $n^2(2\lceil \sqrt{n} \rceil - 1) = \Theta(n^2 \sqrt{n})$ operations.

**Problem 5.**

a) True. $\frac{1}{n^2} \geq \frac{1}{n^2}$ for $n \geq 1$.

b) False. $\lim_{n \to +\infty} \frac{1}{n^a} = n^{b-a} = +\infty$ (since $b > a$).

c) True. 

$$\log(1+n) \leq \log(2n) = 2 \log(\sqrt{2n}) \leq 2\sqrt{2n} = 2\sqrt{2} \cdot \sqrt{n}$$
for $n \geq 1$.

d) True. $\lim_{n \to +\infty} \frac{\sqrt{n}^4}{\sqrt{n}^5} = \lim_{n \to +\infty} n^2 \left(\frac{1}{5}\right)^n = 0$.

e) False. $\lim_{n \to +\infty} \frac{2n^2}{4n^3 \log n} = \lim_{n \to +\infty} 2n^2 - 2n - \log_2(3)n \log n = +\infty$. This means for every $C \geq 0$, there exists $n_0 = n_0(M)$ such that for $n \geq n_0$, $\frac{2n^2}{4n^3 \log n} \geq C$. That is, $2n^2 \geq C4^n3^n \log n$ for $n \geq n_0(C)$. Thus, it is impossible to find $k$ and $C$ such that $2n^2 \leq C4^n3^n \log n$ for $n \geq k$ (since for $n \geq \max\{k, n_0(C)\}$, $2n^2 \geq C4^n3^n \log n$).

f) True. For $n \geq 4$,

$$\frac{n!}{2^n} = \frac{n \times (n-1) \times \cdots \times 4 \times 3 \times 2 \times 1}{2^n} \geq \frac{4 \times 4 \times \cdots \times 4 \times 3 \times 2 \times 1}{2^n} = \frac{4^{n-3} \times 6}{2^n} = \frac{4^n}{2^n} \times \frac{6}{64} = \frac{6}{64} \times 2^n.$$

g) True. For $n \geq 2$,

$$(n!!)^2 = [n \times (n-2) \times (n-4) \times \cdots \times 2] \times [n \times (n-2) \times (n-4) \times \cdots \times 2] \geq [n \times (n-2) \times (n-4) \cdots \times 2] \times [(n-1) \times (n-3) \times (n-5) \times \cdots \times 1] = n!$$

**Problem 6.**

- $100n^3 + n^2$ and $n^2 + n^3$ have the same order.
- $3n^3 + 2^n$ and $n^2 + 2^n$ have the same order.