Discrete Structures — EPFL, Fall 2014

Solution to Problem Set 3

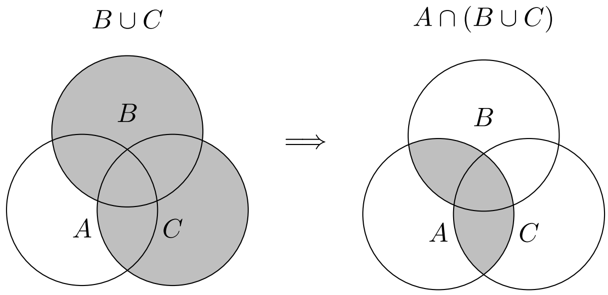
Date: 3.10.2014 — Not graded

Problem 1.

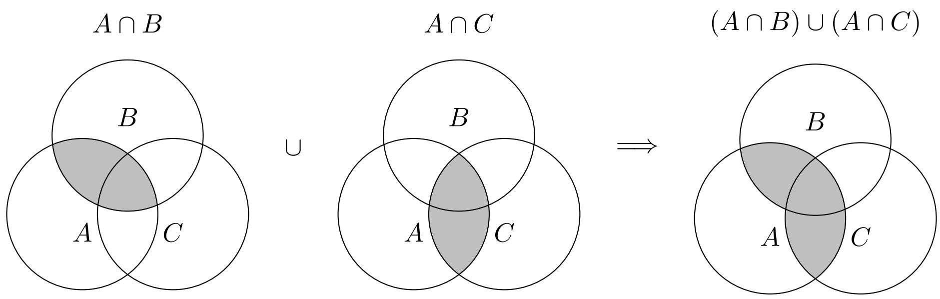
a) We first show $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$. Let $x \in \overline{A \cap B}$. Hence $x \notin A \cap B$ which implies $x \notin A$ or $x \notin B$. Therefore, $x \in \overline{A}$ or $x \in \overline{B}$. Consequently $x \in \overline{A} \cup \overline{B}$.

Reversing the steps shows $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$.

b) Firstly,



On the other side,



Problem 2.

a) True

b) True

c) True

d) False

e) False

f) False

Problem 3.

a) Finite with cardinality 7.

b) Infinite.

c) Finite with cardinality 0 (empty set).

d) Finite with cardinality 0 (empty set).

e) Finite with cardinality 0 (empty set).

f) Infinite.

g) Finite with cardinality 5.

Problem 4. In order for an integer to be multiple of 2, 3, *and* 5 it must be a multiple of $2 \times 3 \times 5 = 30$. There are, hence, 3 numbers in A that are multiple of 2, 3, and 5 (namely, 30, 60, and 90).

To answer the second part of the question, let B, C and D denote the sets of multiples of 2, 3, and 5 respectively. We are looking for $|B \cup C \cup D|$ which is,

$$|B \cup C \cup D| = |B| + |C| + |D| - |B \cap C| - |B \cap D| - |C \cap D| + |B \cap C \cap D|. \qquad (*)$$

Now,

$$|B| = 50, \qquad |C| = 33, \quad |D| = 20,$$

and using the same reasoning above,

$$|B \cap C| = 16, \qquad |B \cap D| = 10, \qquad |C \cap D| = 6$$

(note that $B \cap C$ is the set of multiples of 2 and 3, that is the set of multiples of 6, $B \cap D$ is the set of multiples of 2 and 5, i.e., multiples of 10, and $C \cap D$ is the set of multiples of 3 and 5, i.e. multiples of 15). Finally, as we computed in the first part,

$$|B \cap C \cap D| = 3.$$

Plugging these into (*) we conclude there are 74 integers between 1 and 100 that are multiple of 2, 3, *and* 5.

Problem 5.

a) This is a function but not injective nor surjective.

   The elements a and $-a$ are both mapped to a^{2014} (hence not injective) and no element in $\mathbb{R}$ is mapped to a negative element of $\mathbb{R}$ (hence not surjective).

b) This is not a function (for negative x, $f(x)$ is undefined).

c) This is an injective function but not surjective.

   Suppose $sin(x) = sin(y)$ for some natural numbers $x \neq y$. Then, $x = 2k\pi + y$ or $x = (2k+1)\pi - y$ (for some integer k). But there are now two integers that differ (or sum to) multiples of π. Hence the function is injective.

   The function is not subjective simply because $\sin(x) \in [-1, 1] \subsetneq \mathbb{R}$.

d) This is an injective function but not surjective ($x + 2014$ is always integer hence all the elements of the codomain are not covered).

e) This is an injective and surjective (hence bijective) function.

f) This a surjective but not injective function.

The whole interval of $[-1, 1]$ is covered by $\cos(x)$ but if $x = y + 2\pi$, $\cos(x) = \cos(y)$.

g) This is not a function, the elements $x \in (1, 2)$ are mapped into two different values.

h) This is bijective function.

Indeed, $(1 + x)^2 - (1 - x)^2 = 4x$. Hence $f(x) = 4x$ for $\forall x \in \mathbb{R}$ which is clearly an injective and surjective function.

i) This is not a function; $f(0)$ in undefined.

j) This is an injective but not surjective function.

Suppose $f(x) = f(y)$ for some $x \neq y$. Then $e^{-e^{-x}} = e^{-e^{-y}}$. Hence, $e^{-x} = e^{-y}$ which implies $x = y$. Thus, $f(x)$ is injective.

Also, one can check that $f(x) \in [\frac{1}{2}, \frac{1}{3}]$ thus is not surjective.

Problem 6.

a) $f \circ g = \{(1, 2), (2, 4), (3, 2), (4, 3)\}$.

b) $g \circ f = \{(1, 2), (2, 3), (3, 1), (4, 3)\}$.

c) $g^{-1} = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$.

d) $g \circ g = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$. Note that in the previous part we discovered that $g^{-1} = g$. Thus, $g \circ g = g \circ g^{-1}$ is the identity function on A.

e) f is not injective (both elements 2 and 4 are mapped to 2) hence f^{-1} is undefined.