Exercise 1. Let \( d \geq 1 \) and \( m > 2 \) be integers, and let us consider the following process on \( S = \{0, \ldots, m-1\}^d \): at each step, from state \( x \in S \), pick a component of \( x \) uniformly at random and change it to another number in \( \{0, \ldots, m-1\} \), chosen again uniformly at random.

a) Write down the transition matrix \( P \) of this chain. Is this chain is ergodic? What is its stationary distribution? Is the detailed balance equation satisfied?

It turns out that the eigenvectors of \( P \) are given by \((\phi(z), \ z \in S)\), where
\[
\phi_x(z) = \exp(2\pi ix \cdot z/m), \quad x \in S,
\]
and \( x \cdot z = \sum_{j=1}^{d} x_j z_j \).

b) Compute the corresponding eigenvalues \((\lambda_z, \ z \in S)\) of \( P \).

Hint: Express these in terms of \(|z| = \sharp\{ \text{ non-zero components of } z \}\).

c) Deduce the value of the spectral gap for \( d > 2 \), as well as a corresponding upper bound on \( \| P^m_0 - \pi \|_{TV} \).

d) Compare this upper bound to the general lower bound found in class. Do these two bounds match for large \( m \) and \( d \)?

e) Through a more careful analysis, find a tighter upper bound on \( \| P^m_0 - \pi \|_{TV} \) for large \( m \) and \( d \).

Exercise 2. Regarding the lazy random walk on \( \{0, 1\}^d \), we saw in class that \( \| P^m_0 - \pi \|_{TV} \) is arbitrarily close to 1 for
\[
n = \frac{d+1}{4} (\log d - c)
\]
and \( c > 0 \) arbitrarily large. Following the reasoning made in class (but the technique is simpler here!), show that the following distance:
\[
\| P^m_0 - \pi \|_2 = \left( \sum_{y \in \{0,1\}^d} \left( \frac{p_{0y}(n)}{\sqrt{\pi_y}} - \sqrt{\pi_y} \right)^2 \right)^{1/2}
\]
can be made arbitrarily large by taking again \( n = \frac{d+1}{4} (\log d - c) \) and \( c > 0 \) arbitrarily large.

NB: The above distance is the \( \ell^2 \)-distance between \( P^m_0 \) and \( \pi \); it has been shown in class to be an upper bound on \( \| P^m_0 - \pi \|_{TV} \) (with an extra factor 2).