Exercise 1. A complete graph $K_m$ with $m$ vertices is a graph with one edge between all pairs of vertices. We consider the homogeneous random walk $(X_n, n \geq 0)$ on $K_m$ defined by the transition probability $P(X_{n+1} = j | X_n = i) = 1/(m-1)$ for all vertices $j \neq i$. We also denote by $\mu_i^{(n)} = P(X_n = i), i \in K_m$ the probability distribution of a random walk at time $n$ with initial condition $X_0 = i_0$ and by $\nu_j^{(n)} = P(Y_n = j), j \in K_m$ the probability distribution of another independent random walk with initial condition $Y_0 = j_0$. The two initial vertices are fixed once for all and distinct, $i_0 \neq j_0$.

We now define the following homogeneous Markov process $((X_n, Y_n), n \geq 0)$ with state space $K_m \times K_m$ and transition probabilities

$$P(X_{n+1} = i', Y_{n+1} = j' | X_n = i, Y_n = j) = \begin{cases} 
\frac{1}{1/(m-1)^2} & \text{if } i \neq j \text{ and } i' \neq i, j' \neq j, \\
\frac{1}{m-1} & \text{if } j = i \text{ and } j' = i' \neq i, \\
0 & \text{in all other cases.}
\end{cases}$$

It is understood that this Markov process is conditioned on the initial condition $(X_0, Y_0) = (i_0, j_0)$.

a) Show that $(X_n, Y_n)$ is a coupling of the two probability distributions $\mu^{(n)}$ and $\nu^{(n)}$.

*Hint:* Recall the definition of coupling; you have to compute the marginals of $P(X_n = i, Y_n = j)$.

b) Consider the coalescence time $T = \inf\{n \geq 1 \mid X_n = Y_n\}$ (a random variable). First show that for all $n \geq 1$:

$$P(T > n) = \left(1 - \frac{m-2}{(m-1)^2}\right)^n$$

and deduce from there that for all $n \geq 1$:

$$P(T = n) = \frac{m-2}{(m-1)^2} \left(1 - \frac{m-2}{(m-1)^2}\right)^{n-1}$$

*Hint:* rewrite the events $\{T > n\}$ and $\{T = n\}$ in terms of the $X$’s and $Y$’s.

c) Consider the total variation distance $\|\mu^{(n)} - \nu^{(n)}\|_{TV}$. Use the formula in point 1 of exercise 2 to show that

$$\|\mu^{(n)} - \nu^{(n)}\|_{TV} \leq e^{-n(m-2)/(m-1)^2}$$

*Hint:* $1 - x \leq e^{-x}$ for all $x \geq 0$.

d) We remark that $\pi_i = 1/m, i \in K_m$ is a stationary distribution for the random walk on the complete graph. Justify this remark. Use this remark and the previous bound to deduce

$$\|\mu^{(n)} - \pi\|_{TV} \leq e^{-n(m-2)/(m-1)^2}$$

e) What happens in the (very) particular case $m = 2$?

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Exercise 2. a) Let \( \mu \) and \( \nu \) be two distributions on a state space \( S \) (i.e., \( \mu_i, \nu_i \geq 0 \) for every \( i \in S \) and \( \sum_{i \in S} \mu_i = \sum_{i \in S} \nu_i = 1 \)). Show that the following three definitions of the total variation distance between \( \mu \) and \( \nu \) are equivalent:

1. \( \|\mu - \nu\|_{TV} = \frac{1}{2} \sum_{i \in S} |\mu_i - \nu_i| \).

2. \( \|\mu - \nu\|_{TV} = \max_{A \subset S} |\mu(A) - \nu(A)| \), where \( \mu(A) = \sum_{i \in A} \mu_i \) and \( \nu(A) = \sum_{i \in A} \nu_i \).

3. \( \|\mu - \nu\|_{TV} = \frac{1}{2} \max_{\phi: S \to [-1,1]} |\mu(\phi) - \nu(\phi)| \), where \( \mu(\phi) = \sum_{i \in S} \mu_i \phi_i \) and \( \nu(\phi) = \sum_{i \in S} \nu_i \phi_i \).

Hint: The easiest way is to show that \( 1 \leq 2 \leq 3 \leq 1 \).

b) Show that \( \|\mu - \nu\|_{TV} \) is indeed a distance (i.e., that it is non-negative, that it is zero if and only if \( \mu = \nu \), that it is symmetric and that the triangle inequality is satisfied).