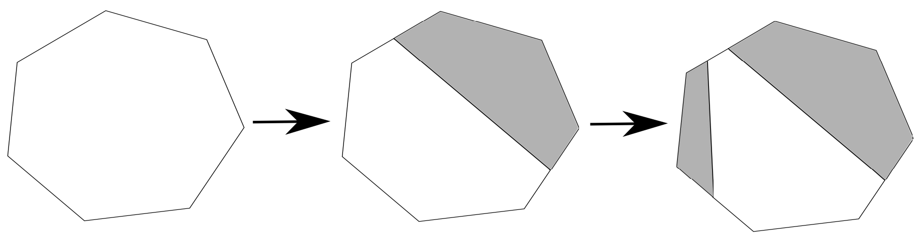
Homework 3 (due Wednesday, October 12)

Exercise 1. A random sequence of convex polygons is generated by picking two edges of the current polygon at random, joining their midpoints, and picking one of the two resulting smaller polygons at random to be the next in the sequence. Let $X_n + 3$ be the number number of edges of the n^{th} polygon thus constructed (so that X_n takes values in the nonnegative integers; for example, $X_n = 0$ corresponds to a triangle, $X_n = 1$ to a quadrilateral, etc).



The white part is selected at each step.

a) What are the transition probabilities associated to the Markov chain $(X_n,\, n \geq 0)$?

b) Compute $\mathbb{E}(X_n)$ in terms of X_0. *Hint*: Compute first $\mathbb{E}(X_n | X_{n-1} = j)$.

c) We define the probability generating function of the process $(X_n, n \geq 0)$ by $G_n(s) = \mathbb{E}(s^{X_n})$. Prove that

$$G_n(s) = \frac{1}{1-s}\,\mathbb{E}\left(\frac{1 - s^{(X_{n-1}+2)}}{X_{n-1}+2}\right) \quad \text{for } n \geq 1$$

d) Now suppose that the process $(X_n, n \geq 0)$ is initialized with $X_0 \sim \pi$, where π is the stationary distribution. Argue that $\mathbb{E}(s^{X_n})$ is independent of n with this initialization. Prove that $G'(s) = G(s)$, $G(1) = 1$, and solve this differential equation.

e) Compute the probability generating function of a Poisson distribution Y with parameter λ, $\mathbb{P}(Y = k) = \lambda^k e^{-\lambda}/k!$. Conclude that the stationary distribution is Poisson.

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Exercise 2. Let $(X_n,\, n \geq 0)$ be a time-homogeneous Markov chain on $S = \{1, 2, 3, \ldots\}$ with transition probabilities:

$$p_{ij} = \mathbb{P}(X_{n+1} = j | X_n = i) = \begin{cases} p_i & \text{if } j = 2i, \\ 1 - p_i, & \text{if } j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

where $0 < p_i < 1$ are arbitrary numbers in general (note in particular that we do *not* ask that $\sum_{i \geq 1} p_i = 1$).

Let us start with the simple case where $p_i \equiv c$ for some $0 < c < 1$.

a) Which states are transient, which are mull-recurrent, which are positive-recurrent? Justify your answer!

b) Does the chain admit a stationary distribution? If yes, compute it!

Assume now that $p_i = 1 - \frac{1}{i+1}$ for $i \geq 1$.

c) Does the chain admit a stationary distribution in this case? If yes, compute it!

Finally, in the general case:

d) Find an optimal condition on the numbers p_i ensuring the existence (and uniqueness) of a stationary distribution.