Exercise 1. Let \((X_n, n \geq 0)\) be an homogeneous Markov chain with transition probabilities

\[ p_{ij}(n) = \mathbb{P}(X_n = j | X_0 = i) \]

We define the probability of first passage as the probability that the chain passes from \(i\) to \(j\) in \(n\) steps without passing by \(j\) before the \(n^{th}\) step.

\[ f_{ij}(n) = \mathbb{P}(X_n = j, X_{n-1} \neq j, \ldots, X_1 \neq j | X_0 = i) \]

We also define the probability of last exit as the probability that the chain passes from \(i\) to \(j\) in \(n\) steps without revisiting \(i\) during these \(n\) steps.

\[ l_{ij}(n) = \mathbb{P}(X_n = j, X_{n-1} \neq i, \ldots, X_1 \neq i | X_0 = i) \]

Let

\[ P_{ij}(s) = \sum_{n=0}^{\infty} p_{ij}(n)s^n, \quad p_{ij}(0) = \delta_{ij} \]

\[ F_{ij}(s) = \sum_{n=0}^{\infty} f_{ij}(n)s^n, \quad f_{ij}(0) = 0 \]

\[ L_{ij}(s) = \sum_{n=0}^{\infty} l_{ij}(n)s^n, \quad l_{ij}(0) = 0 \]

be the associated generating functions. Note that \(L_{ii}(s) = F_{ii}(s)\). Recall that we proved in class that \(P_{ii}(s) = 1 + P_{ii}(s)F_{ii}(s)\).

a) Prove that for \(i \neq j\):

\[ P_{ij}(s) = F_{ij}(s)P_{jj}(s) \]

\[ P_{ij}(s) = P_{ii}(s)L_{ij}(s) \]

b) Deduce the following statements:

1. If \(j\) is recurrent then \(\sum_{n \geq 0} p_{ij}(n) = \infty\) for all \(i\) such that \(f_{ij} > 0\), where \(f_{ij} = \sum_{n \geq 0} f_{ij}(n)\).
2. If \(j\) is transient then \(\sum_{n \geq 0} p_{ij}(n) < \infty\) for all \(i\).
3. If \(j\) is recurrent and \(i\) is transient then \(\sum_{n \geq 0} l_{ij}(n) = \infty\) as long as \(f_{ij} > 0\).

c) Prove that if the Markov chain satisfies \(P_{ii}(s) = P_{jj}(s)\) for all \(i \neq j\), the probability distribution of last exit and first passage are equal.
Exercise 2. Consider the symmetric random walk in 3 dimensions on $\mathbb{Z}^3$ defined during the first lecture:

$$S_0 = (0, 0, 0), \quad S_n = \xi_1 + \ldots + \xi_n, \quad n \geq 1$$

where $(\xi_n, n \geq 1)$ are i.i.d. with

$$\mathbb{P}(\xi_n = e_i) = \mathbb{P}(\xi_n = -e_i) = 1/6$$

and $e_1 = (1, 0, 0), \ e_2 = (0, 1, 0), \ e_3 = (0, 0, 1)$.

a) Argue that

$$\mathbb{P}(S_{2n} = (0, 0, 0)|S_0 = (0, 0, 0)) = \frac{1}{6^{2n}} \sum_{i+j+k=n} \frac{(2n)!}{(i!j!k!)^2}$$

where $i, j, k$ are $\geq 0$.

b) We want to evaluate the asymptotic behaviour of this sum as $n \to \infty$ (we in fact want to derive a good upper bound). Derive the following inequality:

$$\mathbb{P}(S_{2n} = (0, 0, 0)|S_0 = (0, 0, 0)) \leq \left( \frac{1}{2} \right)^{2n} \binom{2n}{n} M \sum_{i+j+k=n} \frac{1}{3^n i! j! k!}$$

where $M = \max\{\frac{n!}{i! j! k!}, \ i + j + k = n, \ i, j, k \geq 0\}$.

c) Next, assuming that the maximum is attained at $i, j, k \approx n/3$, deduce that

$$\mathbb{P}(S_{2n} = (0, 0, 0)|S_0 = (0, 0, 0)) \leq \frac{c}{n^{3/2}}$$

for some constant $c$.

d) Is the random walk in 3 dimensions recurrent or transient?