Homework 1+ (ex.1-2 due Wednesday, September 28)

Exercise 1. Let $(S_n,\ n \in \mathbb{N})$ be the simple asymmetric random walk on $\mathbb{Z}$, defined as

$$S_0 = 0, \quad S_n = \xi_1 + \ldots + \xi_n, \quad n \geq 1,$$

where the random variables $(\xi_n,\ n \geq 1)$ are i.i.d. with $\mathbb{P}(\xi_n = +1) = p \in]0,1[$ and $\mathbb{P}(\xi_n = -1) = q = 1 - p$. Using Stirling's formula (valid for large values of n):

$$n! \sim \sqrt{2\pi n}\ \left(\frac{n}{e}\right)^n,$$

show that

$$p_{0,0}^{(2n)} = \mathbb{P}(S_{2n} = 0 \,|\, S_0 = 0) = \binom{2n}{n} p^n\, q^n \sim \frac{(4pq)^n}{\sqrt{\pi n}}.$$

NB: The notation $a_n \sim b_n$ means precisely

$$\lim_{n \to \infty} \frac{a_n}{b_n} = 1.$$

Exercise 2. Let $\left(\overrightarrow{S_n},\ n \in \mathbb{N}\right)$ be the simple symmetric random walk in two dimensions, that is,

$$\overrightarrow{S_0} = (0,0), \quad \overrightarrow{S_n} = \overrightarrow{\xi_1} + \ldots + \overrightarrow{\xi_n}, \quad n \geq 1,$$

where $\left(\overrightarrow{\xi_n},\ n \geq 1\right)$ are i.i.d random variables such that

$$\mathbb{P}\left(\overrightarrow{\xi_n} = (+1,0)\right) = \mathbb{P}\left(\overrightarrow{\xi_n} = (-1,0)\right) = \mathbb{P}\left(\overrightarrow{\xi_n} = (0,+1)\right) = \mathbb{P}\left(\overrightarrow{\xi_n} = (0,-1)\right) = \frac{1}{4}.$$

Let us write $\overrightarrow{S_n} = (X_n, Y_n)$.

 bf a) What type of (unidimensional) random walks are $(X_n,\ n \in \mathbb{N})$ and $(Y_n,\ n \in \mathbb{N})$?

b) Are these two random walks independent?

Define now $U_n = X_n + Y_n$ and $V_n = X_n - Y_n$, $n \in \mathbb{N}$. Again the same questions:

c) What type of (unidimensional) random walks are $(U_n,\ n \in \mathbb{N})$ and $(V_n,\ n \in \mathbb{N})$?

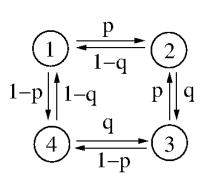**d)** Are these two random walks independent?

e) Deduce from this the value of $\mathbb{P}\left(\overrightarrow{S_{2n}} = (0,0) \,|\, \overrightarrow{S_0} = (0,0)\right)$. How does it behave for large n?

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Important remark. You are *not* supposed to solve completely the following three exercises today, as the questions below refer to the material of the first four weeks of lectures. You should therefore come back to these exercises every week and complete them gradually.

Please pay also attention that even though these three exercises look quite similar, they lead to remarkably different answers.

Exercise 3. Consider the Markov chain $(X_n, n \in \mathbb{N})$ with state space $S = \{1, 2, 3, 4\}$ and the following transition graph:

where $0 \le p, q \le 1$.

a) Write down the transition matrix P of the chain.
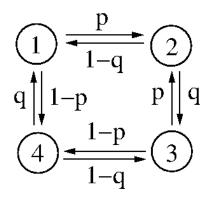
We consider now 3 particular cases:

    1. $p = q = 1$

    2. $p = 1$, $q = 0$

    3. $0 < p, q < 1$

In each case:

b) Describe the equivalence classes of the chain.

c) What is the periodicity of each equivalent class?

d) Which classes are transient / recurrent?

e) Compute the stationary distribution π of the chain. Is it unique?

In case **3** only:

f) Is the stationary distribution also a limiting distribution?

g) Under which condition on the parameters p, q are the detailed balance equations satisfied?

Exercise 4. Consider the Markov chain $(X_n,\, n \in \mathbb{N})$ with state space $S = \{1, 2, 3, 4\}$ and the following transition graph:

where $0 \leq p, q \leq 1$.

a) Write down the transition matrix P of the chain.

We consider now 3 particular cases:

     1. $p = q = 1$

     2. $p = 1,\ q = 0$
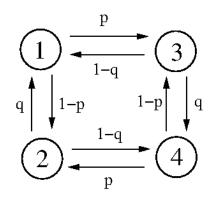
     3. $0 < p, q < 1$

In each case:

b) Describe the equivalence classes of the chain.

c) What is the periodicity of each equivalent class?

d) Which classes are transient / recurrent?

e) Compute the stationary distribution π of the chain. Is it unique?

In case **3** only:

f) Is the stationary distribution also a limiting distribution?

g) Under which condition on the parameters p, q are the detailed balance equations satisfied?

Exercise 5. Consider the Markov chain $(X_n, n \in \mathbb{N})$ with state space $S = \{1, 2, 3, 4\}$ and the following transition graph:

where $0 \le p, q \le 1$.

a) Write down the transition matrix P of the chain.

We consider now 3 particular cases:

    1. $p = q = 1$

    2. $p = 1$, $q = 0$

    3. $0 < p, q < 1$

In each case:

b) Describe the equivalence classes of the chain.

c) What is the periodicity of each equivalent class?

d) Which classes are transient / recurrent?

e) Compute the stationary distribution π of the chain. Is it unique?

In case **3** only:

f) Is the stationary distribution also a limiting distribution?

g) Under which condition on the parameters p, q are the detailed balance equations satisfied?

Final question. In each of the last 3 exercises above, do your answers depend on the labelling of the states 1,2,3,4 ?