Exercises November 7, 2013. Quantum Information Theory and Computation

Exercise 1. Mixtures

a) Show that the two mixtures $\{|0\rangle, \frac{1}{2}; |1\rangle, \frac{1}{2}\}$ and $\{\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle), \frac{1}{2}; \frac{1}{\sqrt{2}}(|0\rangle-|1\rangle), \frac{1}{2}\}$ have the same density matrix.

b) Consider the mixture $\{|0\rangle, \frac{1}{2}; \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \frac{1}{2}\}$. Give the spectral decomposition of the density matrix.

Exercise 2. Intermediate state in teleportation

a) In the teleportation protocol, before the measurements of Alice, what are the density matrices that Bob (resp. Alice) should use to describe their state? What is the associated entropy?

b) After Alice's measurement, what are the density matrices that Bob (resp. alice) should use to describe their state (it is assumed Bob has no information on Alice's measurement, but that Alice knows her measurement)? What is the associated entropy?

c) After the classical communication phase (Alice sends 2 bits to Bob) what are the density matrices that Bob (resp. Alice) should use to describe their state? What is the associated entropy?

Exercise 3. Reduced density matrix

a) Take the first GHZ state for three Qbits $\frac{1}{\sqrt{2}}(|000\rangle+|111\rangle)$ in the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$. Compute the reduced density matrices ρ_{AB} and ρ_C.

b) Take the state $|\Phi\rangle \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ used in teleportation. Compute Alice's and Bob's reduced density matrices.

c) Check that the Schmidt decomposition theorem holds in each of the above cases.

Exercise 4. Schmidt decomposition theorem

Consider the pure N Qbit state,

$$|\Psi\rangle = \frac{1}{2^{N/2}} \sum_{b_1 \dots b_N \in \{0,1\}^N} |b_1 \dots b_N\rangle$$

a) Compute the density matrix of the first Qbit. Show that it has non degenerate eigenvalues 0 and 1.

b) Compute the reduced density matrix of the set of bits $(2...N)$. Show that this $2^{N-1} \times 2^{N-1}$ matrix has a non degenerate eigenvalue 1 and an eigenvalue 0 with degeneracy $2^{N-1} - 1$.

c) Check explicitly that the Schmidt decomposition theorem holds.

Exercise 5. Remarks about purification

Consider a truly mixed state: one that is not extremal in the convex set of density matrices. Is it possible to purify with a pure tensor product state? Is the purification unique?