
We consider a source of mixed states $\rho_x$ occurring each with probabilities $p_x$. Messages are $N$ letter strings of the form $\rho_{x_1} \otimes \ldots \otimes \rho_{x_N}$ and have a probability $p_{x_1} \ldots p_{x_N}$. In this exercise we want to give some support to the conjecture that the achievable rate of compression for a source of mixed states is equal to the Holevo quantity

$$\chi(\{p_x, \rho_x\}) = S(\rho) - \sum_x p_x S(\rho_x), \quad \rho = \sum_x p_x \rho_x$$

Note that in the case of a source of pure states $\rho_x = |\phi_x\rangle\langle \phi_x|$ the Holevo quantity $\chi(\rho)$ reduce to $S(\rho)$ which is the optimal achievable rate given by Schumacher’s theorem.

a) Take a source constituted of the unique letter $\rho_0$ occurring with probability $p_0 = 1$. How many bits are needed to compress this source? What is the value of $\chi(\rho)$? Is this consistent?

b) Now consider a source of mixed mutually orthogonal states. Two mixed states are said to be mutually orthogonal if

$$\text{Tr} \rho_x \rho_y = 0, \quad x \neq y$$

Construct purifications $|\Psi_x\rangle$ of $\rho_x$ that satisfy (hint: use the spectral decomposition)

$$\langle \Psi_x | \Psi_y \rangle = 0, \quad x \neq y$$

What would be an encoding scheme achieving a compression rate of $H(X) = -\sum_x p_x \log p_x$? Why would this rate be optimal? Check that in the present case we have

$$H(X) = \chi(\rho)$$

hint: no big calculations.