Problem 1. Note that $E_0 = E_1 \cup E_2 \cup E_3$.

(a) (1) For disjoint events, $P(E_0) = P(E_1) + P(E_2) + P(E_3)$, so $P(E_0) = 3/4$.

(2) For independent events, $1 - P(E_0)$ is the probability that none of the events occur, which is the product of the probabilities that each one doesn’t occur. Thus $1 - P(E_0) = (3/4)^3$ and $P(E_0) = 37/64$.

(3) If $E_1 = E_2 = E_3$, then $E_0 = E_1$ and $P(E_0) = 1/4$.

(b) (1) From the Venn diagram in Fig. 1, $P(E_0)$ is clearly maximized when the events are disjoint, so $\max P(E_0) = 3/4$.

![Venn Diagram for problem 1 (b)(1)](image1)

(2) The intersection of each pair of sets has probability 1/16. As seen in Fig. 2, $P(E_0)$ is maximized if all these pairwise intersections are identical, in which case $P(E_0) = 3 \left(1/4 - 1/16\right) + 1/16 = 5/8$. One can also use the formula $P(E_0) = P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_1 \cap E_3) - P(E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_3)$, and notice that all the terms except the last is fixed by the problem, and the last term cannot be made more than $\min_{i,j} P(E_i \cap E_j) = 1/16$.

![Venn Diagram for problem 1 (b)(2)](image2)

(c) Same considerations as in (b)(2) yields the upper bound $P(E_0) \leq 3p - 2p^2$ As $P(E_0) = 1$, we find that $p \geq 1/2$. 

Problem 2. Let $L$ be the event that the loaded die is picked and $H$ the event that the honest die is picked. Let $A_i$ be the event that $i$ is turned up on the first roll, and $B_i$ be the event that $i$ is turned up on the second roll. We are given that $P(L) = 1/3$, $P(H) = 2/3$; $P(A_i \mid L) = 2/3$; $P(A_i \mid L) = 1/15$ for $2 \leq i \leq 6$; $P(A_i \mid H) = 1/6$ for $1 \leq i \leq 6$. Then

$$P(L \mid A_1) = \frac{P(L, A_1)}{P(A_1)} = \frac{P(A_1 \mid L) P(L)}{P(A_1 \mid L) P(L) + P(A_1 \mid H) P(H)} = \frac{2}{3}.$$ 

This is the probability that the loaded die was picked conditional on the first roll showing a 1. For two rolls we make the assumption from the physical mechanism involved in rolling a die that the outcome on the two successive rolls of a given die are independent. Thus $P(A_1B_1 \mid L) = (2/3)^2$ and $P(A_1B_1 \mid H) = (1/6)^2$. It follows as before that

$$P(L \mid A_1B_1) = \frac{8}{9}.$$ 

Problem 3. Since $A, B, C, D$ form a Markov chain their probability distribution is given by

$$p(a)p(b\mid a)p(c\mid b)p(d\mid c) \tag{1}$$

(a) Yes: Summing (1) over $d$ shows that $A, B, C$ have the probability distribution $p(a)p(b\mid a)p(c\mid b)$.

(b) Yes: The reverse of a Markov chain is also a Markov chain. Applying this to $A, B, C, D$ and using part (a) we get that $D, C, B$ is a Markov chain. Reversing again we get the desired result.

(c) Yes: Since $A, B, C, D$ is a Markov chain, given $C, D$ is independent of $B$, and thus $p(d\mid c) = p(d\mid b, c))$. So (1) can be written as

$$p(a, (b, c), d) = p(a)p((b, c)\mid a)p(d\mid (b, c)).$$

(d) Yes, by a similar (in fact easier) reasoning as (c).

Problem 4. No. Take for example $A = D$ and let $A$ be independent of the pair $(B, C)$. Then both $A, B, C$ and $B, C, A$ (same as $B, C, D$) are Markov chains. But $A, B, C, D$ is not: $A$ is not independent of $D$ when $B$ and $C$ are given.

Problem 5.

(a)

$$E[X + Y] = \sum_{x,y} (x + y)P_{XY}(x, y)$$

$$= \sum_{x,y} xP_{XY}(x, y) + \sum_{x,y} yP_{XY}(x, y)$$

$$= \sum_{x} xP_{X}(x) + \sum_{y} yP_{Y}(y)$$

$$= E[X] + E[Y].$$

Note that independence is not necessary here and that the argument extends to non-discrete variables if the expectation exists.
E[XY] = \sum_{x,y} xy P_{XY}(x, y) \\
= \sum_{x,y} xy P_X(x) P_Y(y) \\
= \sum_x x P_X(x) \sum_y y P_Y(y) \\
= E[X] E[Y].

Note that the statistical independence was used on the second line. Let X and Y take on only the values ±1 and 0. An example of uncorrelated but dependent variables is

\[ P_{XY}(1, 0) = P_{XY}(0, 1) = P_{XY}(-1, 0) = P_{XY}(0, -1) = \frac{1}{4}. \]

An example of correlated and dependent variables is

\[ P_{XY}(1, 1) = P_{XY}(-1, -1) = \frac{1}{2}. \]

(c) Using (a), we have

\[ \sigma^2_{X+Y} = E[(X - E[X] + Y - E[Y])^2] \]

\[ = E[(X - E[X])^2] + 2E[(X - E[X])(Y - E[Y])] + E[(Y - E[Y])^2]. \]

The middle term, from (a), is 2(E[XY] - E[X]E[Y]). For uncorrelated variables that is zero, leaving us with \( \sigma^2_{X+Y} = \sigma_X^2 + \sigma_Y^2 \).

PROBLEM 6. We solve the problem for a general vehicle with \( n \) wheels.

(a) Out of \( n! \) possible orderings \((n - 1)!\) has the tyre 1 in its original place. Thus tyre 1 is installed in its original position with probability \( 1/n \).

(b) All tyres end up in their original position in only 1 of the \( n! \) orders. Thus the probability of this event is \( 1/n! \).

(c) Let \( X_i \) be the indicator random variable that tyre \( i \) is installed in its original position, so that the number of tyres installed in their original positions is \( N = \sum_{i=1}^{n} X_i \). By (a), \( E[X_i] = 1/n \). By the linearity of expectation, \( E[N] = n(1/n) = 1 \). Note that the linearity of the expectation holds even if the \( X_i \)'s are not independent (as it is in this case).

(e) Let \( A_i \) be the event that the \( i \)th tyre remains in its original position. Then, the event we are interested in is the complement of the event \( \bigcup_i A_i \) and thus has probability \( 1 - \Pr(\bigcup_i A_i) \). Furthermore, by the inclusion/exclusion formula,

\[ \Pr(\bigcup_i A_i) = \sum_i \Pr(A_i) - \sum_{i_1 < i_2} \Pr(A_{i_1} \cap A_{i_2}) + \sum_{i_1 < i_2 < i_3} P(A_{i_1} \cap A_{i_2} \cap A_{i_3}) - \ldots. \]

The \( j \)th sum above consists of \( \binom{n}{j} \) terms, each term having the value \( P(A_1 \cap \cdots \cap A_j) \). Note that this is the probability of the event that tyres 1 through \( j \) have remained in their original positions, and equals \( (n-j)!/n! \). Consequently,

\[ \Pr(\bigcup_i A_i) = \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} (n-j)!/n! = \sum_{j=1}^{n} (-1)^{j-1} 1/j!, \]
and the event that no tyre remains in its original position has probability

$$1 - \Pr\left(\bigcup_i A_i\right) = \sum_{j=0}^{n} \frac{(-1)^j}{j!}.$$

(For the case $n = 4$, the value is $3/8$.)

**Problem 7.**

(a) Let $A_i$ denote the event that $X_i \neq X$. The event that $X$ does not appear in the inventory is thus

$$A = A_1 \cap \ldots A_n.$$  

Note that the events $A_1, \ldots, A_n$ are not independent—because they involve the common random variable $X$. However, they become independent when conditioned on the value of $X$, with $P(A_i|X = x) = 1 - p(x)$. Thus,

$$P(A|X = x) = (1 - p(x))^n.$$  

Consequently $P(A) = \sum_x p(x)(1 - p(x))^n$.

(b) With $p$ the uniform distribution on $n$ items, the above value for $P(A)$ equals $(1 - 1/n)^n$.

(c) For $n$ large, $(1 - 1/n)^n$ approaches $1/e \approx 37\%$. 
