Problem 1. Let \( p_{XY}(x,y) \) be given by

\[
\begin{array}{c|cc}
  & 0 & 1 \\
\hline
 0 & 1/3 & 1/3 \\
 1 & 0 & 1/3
\end{array}
\]

Find

(a) \( H(X) \), \( H(Y) \).
(b) \( H(X|Y) \), \( H(Y|X) \).
(c) \( H(X,Y) \).
(d) \( H(Y) - H(Y|X) \).
(e) \( I(X;Y) \).
(f) Draw a Venn diagram for the quantities in (a) through (e).

Problem 2. Let \( X \) be a random variable taking values in \( M \) points \( a_1, \ldots, a_M \), and let \( P_X(a_M) = \alpha \). Show that

\[
H(X) = \alpha \log \frac{1}{\alpha} + (1 - \alpha) \log \frac{1}{1 - \alpha} + (1 - \alpha) H(Y)
\]

where \( Y \) is a random variable taking values in \( M - 1 \) points \( a_1, \ldots, a_{M-1} \) with probabilities \( P_Y(a_j) = P_X(a_j)/(1 - \alpha) \); \( 1 \leq j \leq M - 1 \). Show that

\[
H(X) \leq \alpha \log \frac{1}{\alpha} + (1 - \alpha) \log \frac{1}{1 - \alpha} + (1 - \alpha) \log(M - 1)
\]

and determine the condition for equality.

Problem 3. Let \( X, Y, Z \) be discrete random variables. Prove the validity of the following inequalities and find the conditions for equality:

(a) \( I(X,Y;Z) \geq I(X;Z) \).
(b) \( H(X,Y|Z) \geq H(X|Z) \).
(c) \( H(X,Y,Z) - H(X,Y) \leq H(X,Z) - H(X) \).
(d) \( I(X;Z|Y) \geq I(Z;Y|X) - I(Z;Y) + I(X;Z) \).
PROBLEM 4. For a stationary process \(X_1, X_2, \ldots\), show that

(a) \(\frac{1}{n} H(X_1, \ldots, X_n) \leq \frac{1}{n-1} H(X_1, \ldots, X_{n-1}).\)

(b) \(\frac{1}{n} H(X_1, \ldots, X_n) \geq H(X_n|X_{n-1}, \ldots, X_1).\)

PROBLEM 5. Let \(\{X_i\}_{i=-\infty}^{\infty}\) be a stationary stochastic process. Prove that

\(H(X_0|X_{-1}, \ldots, X_{-n}) = H(X_0|X_1, \ldots, X_n).\)

That is: the conditional entropy of the present given the past is equal to the conditional entropy of the present given the future.

PROBLEM 6. Show, for a Markov chain, that

\(H(X_0|X_n) \geq H(X_0|X_{n-1}), \quad n \geq 1.\)

Thus, initial state \(X_0\) becomes more difficult to recover as time goes by.

PROBLEM 7. Let \(X_1, X_2, \ldots\) be i.i.d., each with probability distribution \(p(x)\). Show that with probability one

\(\lim_{n \to \infty} p(X_1, \ldots, X_n)^{1/n}\)

exists, and find its value. Hint: use the AEP.

Compare this to

\(\lim_{n \to \infty} E[p(X_1, \ldots, X_n)^{1/n}].\)