Graph Theory Applications | EPFL, Spring 2014

Midterm Exam

Date: 17.04.2014

Rules:

- This exam is closed book. No electronic items are allowed. You are only allowed to have two handwritten single-sided A4 pages of notes. Place all your personal items on the floor. Leave only a pen and your ID on the desk. If you need extra scratch paper, please ask for it by raising your hand.
- Please do not cheat. We will be forced to report any such occurrence to the president of EPFL. This is not how you want to meet him. :-(
- The exam starts at 16:15 and lasts till 18:00.
- If a question is not completely clear to you don't waste time and ask us for clarification right away.
- It is not necessarily expected that you solve all problems. Don't get stuck. Start with the problem which seems the easiest to you and try to collect as many points as you can.

Name : ....

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Problem 1 ... / 20

Problem 2 ... / 20

Problem 3 ... / 20

Problem 4 ... / 20

TOTAL ... / 80

(scratch paper)

Problem 1. *[20points]* In class we discussed the concept of a *matroid* and we saw how it naturally appears when analyzing greedy algorithms. Recall that a matroid $\mathcal{M}$ is a pair $(E, \mathcal{I})$, where E is the set of elements and $\mathcal{I}$ is the set of *independent* subsets of E with the following properties.

- $\emptyset \in \mathcal{I}$
- If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$.
- If $A, B \in \mathcal{I}$ and $|A| < |B|$, then there exists an element $e \in B$ so that $A \cup \{e\} \in \mathcal{I}$.

Consider a bipartite graph $G = (E, V)$ with bipartition $V = X \cup Y$. Let $\mathcal{I}_X$ be the set of subsets of edges E so that the induced degrees on X are either 0 or 1. In other words, if $I \in \mathcal{I}_X$, then the subgraph of G induced by I is s.t. the degrees of the vertices in X are either 0 or 1. In the same manner define $\mathcal{I}_Y$.

(i) *[10points]* Prove that $\mathcal{M}_X = (E, \mathcal{I}_X)$ and $\mathcal{M}_Y = (E, \mathcal{I}_Y)$ are matroids.

(ii) *[10points]* Show that a subset M of the set of edges is a matching if and only if $M \in \mathcal{I}_X \cap \mathcal{I}_Y$.

Remark: As we see, matchings are representable as the intersection of matroids. This representation gives rise to efficient algorithms to find maximum and optimum matchings.

(scratch paper)

Problem 2. *[20points]* Let $G = (E, V)$ be a weighted graph with weights $w : E \to \mathbb{R}$. Recall that an MST (minimum spanning tree) is a spanning tree T so that the sum of its weights, $\sum_{e \in T} w(e)$, is minimized.

A minimum bottleneck spanning tree (MBST) is a spanning tree so that the maximum weight of all included edges is minimized, i.e., $\max_{e \in T} w(e)$ is minimum.

(i) *[10points]* Show that an MST is an MBST.

(ii) *[10points]* Show that the converse is in general false by giving an example of an MBST which is not an MST.

(scratch paper)

Problem 3. *[20points]* The *Boolean lattice* BL_n is the graph $G = (E, V)$, where $V = \mathcal{P}(\{1, 2, \cdots, n\})$ (set of all subsets of $\{1, 2, \cdots, n\}$) and two vertices (subsets) are adjacent if the associated subsets have a symmetric difference (union minus intersection) of cardinality 1.

Prove that BL_n is bipartite.

(scratch paper)

Problem 4. *[20points]* Consider the graph $G(V, E)$ with adjacency matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

(Note that this graph one vertex has a self loop.)

This matrix has a maximum eigenvalue of $\lambda = 1 + \sqrt{2}$ and an associated eigenvector $x^T = (1, 1/\sqrt{2}, 1/\sqrt{2})$.

Give a good upper and lower bound on the number of walks of length n from vertex 1 back to vertex 1.

Note: *Good* here means that the ratio of the two bounds is $\Theta(1)$, i.e., it stays constant as n grows large.

(scratch paper)