Problem 1. We will show a constructive procedure to obtain a path from \( u \) to \( v \), given a walk from \( u \) to \( v \). Let \( V_{rep} \subseteq V(G) \) be the set of vertices that appear in the walk at least two times. If \( V_{rep} = \emptyset \), then each vertex appears in the walk at most once, which means that the walk is already a path. Suppose \( V_{rep} \neq \emptyset \) and pick \( v \in V_{rep} \). Remove from the walk all the vertices and edges which appear between the first and the last occurrence of \( v \), including the first occurrence of \( v \). In this way, we obtain a walk in which the vertex \( v \) appears only once. Repeat the same procedure for all the elements of \( V_{rep} \) to obtain a path from \( u \) to \( v \).

Problem 2. The proof is by induction on the length \( k \) of the walk.

Base step: \( k = 1 \). The number of walks from \( v_i \) to \( v_j \) is 1 if the two vertices are connected, and 0 otherwise. The \((i, j)\)-th entry of \( A \) is 1 if \( v_i \) and \( v_j \) are connected, and 0 otherwise. This suffices to prove the claim for \( k = 1 \).

Induction step. A walk from \( v_i \) to \( v_j \) of length \( k \) is formed by a walk from \( v_i \) to \( v_m \) of length \( k - 1 \) and by a walk from \( v_m \) to \( v_j \) of length 1 for some \( v_m \in V \). Then,

\[
\text{# walks from } v_i \text{ to } v_j = \sum_{v_m \in V} (\text{# walks from } v_i \text{ to } v_m) \cdot (\text{# walks from } v_m \text{ to } v_j).
\]

By the induction hypothesis, the number of walks from \( v_i \) to \( v_m \) of length \( k - 1 \) is the \((i, m)\)-th entry of \( A^{k-1} \), namely \( a_{i,m}^{(k-1)} \), and the number of walks from \( v_m \) to \( v_j \) of length 1 is the \((m, j)\)-th entry of \( A \), namely \( a_{m,j}^{(1)} \). Hence, the number of walks from \( v_i \) to \( v_j \) of length \( k \) is \( \sum_m a_{i,m}^{(k-1)} \cdot a_{m,j}^{(1)} = a_{i,j}^{(k)} \), which is the \((i,j)\)-th entry of \( A^k \).

Problem 3. Suppose that the thesis is false, i.e., there exists two longest paths \( P_1 \) and \( P_2 \) which do not share any vertex. Denote by \( L \) the length of these longest paths and let \( u_1, u_2, \ldots, u_L \) and \( v_1, v_2, \ldots, v_L \) be the vertices of \( P_1 \) and \( P_2 \), respectively. First, note that there exists a path \( \tilde{P} \) from one vertex of \( P_1 \), call it \( u_i \), to one vertex of \( P_2 \), call it \( v_j \), which does not contain any other vertex of \( P_1 \) or \( P_2 \). To see this, let us consider a path from \( u_1 \) to \( v_1 \). Such a path exists because the graph is connected. If this path does not contain any other vertex of \( P_1 \) or \( P_2 \), we are done. Otherwise, starting from \( u_1 \) toward \( v_1 \), let \( u_i \) and \( v_j \) be the last vertex of \( P_1 \) and \( P_2 \), respectively, met on that path.

Suppose that \( i \geq L/2 \) and \( j \geq L/2 \). Consider the path \( P_{\max} \) which goes from \( u_1 \) to \( v_1 \) and is formed by the following three parts:

1. the part of \( P_1 \) which goes from \( u_1 \) to \( u_i \);
2. the path \( \tilde{P} \);
3. the part of \( P_2 \) which goes from \( v_j \) to \( v_1 \).

The length of \( P_{\max} \) is at least \( L/2 + 1 + L/2 = L + 1 \), contradicting the fact that the longest path has length \( L \). If \( i < L/2 \), replace \( u_1 \) with \( u_L \) in the previous construction and, if \( j < L/2 \), replace \( v_1 \) with \( v_L \).

Problem 4. We will use induction on the number of vertices in the graph. Clearly the statement holds for \( n = 2 \). Assume the statement is true for a tournament with \( n \) vertices and consider a tournament on
Let $G'$ be the graph we get from $G$ by taking out one of the vertices, say, $v_{k+1}$ (and all of its adjacent edges). Clearly, $G'$ is also a tournament and by the induction hypothesis, it has a directed Hamiltonian path, say $v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k$. Now look at $G$: if there is an edge directed from $v_{k+1}$ to $v_1$ or from $v_k$ to $v_{k+1}$ we are done (just extend the path in $G'$ to a path in $G$). Otherwise, there is an edge from $v_1$ to $v_{k+1}$ and from $v_{k+1}$ to $v_k$. There are three possibilities:

(a) all edges from $v_{k+1}$ to $v_i$, $2 \leq i \leq k - 1$ are directed from $v_{k+1}$ to $v_i$; then the Hamilton path is $v_1 \rightarrow v_{k+1} \rightarrow v_2 \ldots v_k$;

(b) all edges from $v_{k+1}$ to $v_i$, $2 \leq i \leq k - 1$ are directed from $v_i$ to $v_{k+1}$; then the Hamilton path is $v_1 \rightarrow v_2 \ldots v_{k-1} \rightarrow v_{k+1} \rightarrow v_k$;

(c) let $1 \leq i \leq k - 1$ be the smallest index such that there is an edge directed from $v_i$ to $v_{k+1}$ and an edge directed from $v_{k+1}$ to $v_{i+1}$. (There must be such an $i$ otherwise we would be in case (a) or (b)). Then in the path in $G'$, replace $(v_i \rightarrow v_{i+1})$ by $v_i \rightarrow v_{k+1} \rightarrow v_{i+1}$ to get a Hamiltonian path in $G$.

**Problem 5.** Let $L = \{v_1, v_2, \ldots, v_l\}$ be the longest path in $G$ (in the given order). First we show that $l \geq k + 1$. If $l \leq k$ then consider all the neighbors of the vertex $v_l$. By assumption, $v_l$ is of degree $k$ or larger. This means that $v_l$ has a neighbor other than $v_1, v_2, \ldots, v_{l-1}$. But in this case we can extend the path by 1 by including this neighbor, contradicting the maximality of $L$. Now consider the vertex $v_1$. By assumption it is connected to at least $k$ vertices. Since $L$ is the longest path in $G$, all of the neighbors of $v_1$ belong to this path. Further, since $v_1$ has degree $\geq k$, one of its neighbors $v_l$ has to be from the set $\{v_{k+1}, \ldots, v_l\}$. Then $v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_l \rightarrow v_1$ forms a cycle of length $\geq k + 1$. 

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