Graph Theory Applications EPFL, Spring 2014

Solution to Problem Set 2

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Problem 1. We will show a constructive procedure to obtain a path from u to v, given a walk from u to v. Let $V_{\text{rep}} \subset V(G)$ be the set of vertices that appear in the walk at least two times. If $V_{\text{rep}} = \emptyset$, then each vertex appears in the walk at most once, which means that the walk is already a path. Suppose $V_{\text{rep}} \neq \emptyset$ and pick $\bar{v} \in V_{\text{rep}}$. Remove from the walk all the vertices and edges which appear between the first and the last occurrence of $\bar{v}$, including the first occurrence of $\bar{v}$. In this way, we obtain a walk in which the vertex $\bar{v}$ appears only once. Repeat the same procedure for all the elements of V_{rep} to obtain a path from u to v.

Problem 2. The proof is by induction on the length k of the walk.

Base step: $k = 1$. The number of walks from v_i to v_j is 1 if the two vertices are connected, and 0 otherwise. The (i, j)-th entry of A is 1 if v_i and v_j are connected, and 0 otherwise. This suffices to prove the claim for $k = 1$.

Induction step. A walk from v_i to v_j of length k is formed by a walk from v_i to v_m of length $k - 1$ and by a walk from v_m to v_j of length 1 for some $v_m \in V$. Then,

$$\#\text{ walks from } v_i \text{ to } v_j = \sum_{v_m \in V} (\#\text{ walks from } v_i \text{ to } v_m) \cdot (\#\text{ walks from } v_m \text{ to } v_j).$$

By the induction hypothesis, the number of walks from v_i to v_m of length $k - 1$ is the (i, m)-th entry of A^{k-1}, namely $a_{i,m}^{(k-1)}$, and the number of walks from v_m to v_j of length 1 is the (m, j)-th entry of A, namely $a_{m,j}^{(1)}$. Hence, the number of walks from v_i to v_j of length k is $\sum_m a_{i,m}^{(k-1)} \cdot a_{m,j}^{(1)} = a_{i,j}^{(k)}$, which is the (i, j)-th entry of A^k.

Problem 3. Suppose that the thesis is false, i.e., there exists two longest paths P_1 and P_2 which do not share any vertex. Denote by L the length of these longest paths and let $u_1, u_2, \cdots, u_L$ and $v_1, v_2, \cdots, v_L$ be the vertices of P_1 and P_2, respectively. First, note that there exists a path $\bar{P}$ from one vertex of P_1, call it u_i, to one vertex of P_2, call it v_j, which does not contain any other vertex of P_1 or P_2. To see this, let us consider a path from u_1 to v_1. Such a path exists because the graph is connected. If this path does not contain any other vertex of P_1 or P_2, we are done. Otherwise, starting from u_1 toward v_1, let u_i and v_j be the last vertex of P_1 and P_2, respectively, met on that path.

Suppose that $i \geq L/2$ and $j \geq L/2$. Consider the path P_{max} which goes from u_1 to v_1 and is formed by the following three parts:

1. the part of P_1 which goes from u_1 to u_i;
2. the path $\bar{P}$;
3. the part of P_2 which goes from v_j to v_1.

The length of P_{max} is at least $L/2 + 1 + L/2 = L + 1$, contradicting the fact that the longest path has length L. If $i < L/2$, replace u_1 with u_L in the previous construction and, if $j < L/2$, replace v_1 with v_L.

Problem 4. We will use induction on the number of vertices in the graph. Clearly the statement holds for $n = 2$. Assume the statement is true for a tournament with n vertices and consider a tournament on

$n + 1$ vertices. Let G' be the graph we get from G by taking out one of the vertices, say, v_{k+1} (and all of its adjacent edges). Clearly, G' is also a tournament and by the induction hypothesis, it has a directed Hamiltonian path, say $v_1 \to v_2 \to \ldots \to v_k$. Now look at G: if there is an edge directed from v_{k+1} to v_1 or from v_k to v_{k+1} we are done (just extend the path in G' to a path in G). Otherwise, there is an edge from v_1 to v_{k+1} and from v_{k+1} to v_k. There are three possibilities:
(a) all edges from v_{k+1} to v_i, $2 \leq i \leq k-1$ are directed from v_{k+1} to v_i; then the Hamilton path is $v_1 \to v_{k+1} \to v_2 \ldots v_k$;
(b) all edges from v_{k+1} to v_i, $2 \leq i \leq k-1$ are directed from v_i to v_{k+1}; then the Hamilton path is $v_1 \to v_2 \ldots v_{k-1} \to v_{k+1} \to v_k$;
(c) let $1 \leq i \leq k-1$ be the smallest index such that there is an edge directed from v_i to v_{k+1} and an edge directed from v_{k+1} to v_{i+1}. (There must be such an i otherwise we would be in case (a) or (b)). Then in the path in G', replace $(v_i \to v_{i+1})$ by $v_i \to v_{k+1} \to v_{i+1}$ to get a Hamiltonian path in G.

Problem 5. Let $L = \{v_1, v_2, \ldots, v_l\}$ be the longest path in G (in the given order). First we show that $l \geq k+1$. If $l \leq k$ then consider all the neighbors of the vertex v_l. By assumption, v_l is of degree k or larger. This means that v_l has a neighbor other than $v_1, v_2, \ldots, v_{l-1}$. But in this case we can extend the path by 1 by including this neighbor, contradicting the maximality of L. Now consider the vertex v_1. By assumption it is connected to at least k vertices. Since L is the longest path in G, all of the neighbors of v_1 belong to this path. Further, since v_1 has degree $\geq k$, one of its neighbors v_t has to be from the set $\{v_{k+1}, \ldots, v_l\}$. Then $v_1 \to v_2 \to \ldots \to v_t \to v_1$ forms a cycle of length $\geq k+1$.