Problem 1. Suppose that \( n \) people are attending a party and there are some handshakes between different people in the party. Show that there are at least two persons who have shaken hands with the same number of people.

*Hint. Pigeonhole.*

Problem 2. Balls of 8 different colors are placed in 6 jars. There are 20 balls of each color. Show that there must be a jar containing two pairs from two different colors of balls (for example, there is a jar containing at least two blue and at least two green balls).

Problem 3. Prove that for any \( n \in \mathbb{N} \),

\[
\sum_{i=0}^{n} \binom{n}{i} = 2^{n-1},
\]

where \( \binom{n}{i} = \frac{n!}{i!(n-i)!} \) denotes the binomial coefficient.

*Hint. There is a lot of ways in which one can prove the claim. One way is to show first that \( \sum_{i=0}^{n}(-1)^i \binom{n}{i} = 0 \) with the binomial theorem.*

Problem 4. Let \( d = (d_1, d_2, \cdots, d_n) \) be a nonincreasing sequence of non-negative integers. We say that \( d \) is graphic if there exists a simple graph with degree sequence \( d \). Recall that in class we discussed that for the sequence \( d \) to be graphic we need that

\[
\sum_{i=1}^{n} d_i \text{ is even,}
\]

\[
\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min(k, d_i), \quad 1 \leq k \leq n.
\]

(1)

The aim of this exercise is to show an algorithm to construct a graph with degree sequence \( d \), if such a graph exists.

1. Suppose that \( d \) is graphic, i.e., there exists a graph \( G \) which has degree sequence \( d \). Show that there exists a graph \( G' \) s.t. the vertex with degree \( d_1 \) is connected to the vertices with degrees \( d_2, d_3, \cdots, d_{d_1+1} \).

2. Let \( d' = (d_2 - 1, d_3 - 1, \cdots, d_{d_1+1} - 1, d_{d_2+d_1}, \cdots, d_n) \). Use the previous result to show that \( d' \) is graphic if and only if \( d' \) is graphic.

3. Now, what could be an algorithm which constructs a graph with degree sequence \( d \) (if such a graph exists)?

Note that with the procedure outlined in this exercise we can prove that the conditions (1) are also sufficient for the sequence \( d \) to be graphic.
For the next two problem we are going to need some definitions. Let \(d(u, v)\) denote the graph distance between the vertices \(u\) and \(v\) \((u, v \in V(G))\), which is the minimum length of the paths connecting them, i.e., the length of the shortest path.

Let \(\text{diam}(G)\) denote the diameter of \(G\), which is the longest shortest path between any two vertices of the graphs, i.e.,

\[
\text{diam}(G) = \max_{u, v \in V(G)} d(u, v).
\]

Let \(\text{ecc}(v)\) denote the eccentricity of the vertex \(v \in V(G)\), which is defined as the maximum graph distance between \(v\) and any other vertex \(u \in V(G)\), i.e.,

\[
\text{ecc}(v) = \max_{u \in V(G)} d(u, v).
\]

Let \(\text{rad}(G)\) denote the radius of the graph \(G\), which is defined as the minimum graph eccentricity of any graph vertex in \(G\), i.e.,

\[
\text{rad}(G) = \min_{v \in V(G)} \text{ecc}(v).
\]

Let \(N(v)\) denote the neighborhood of a vertex \(v \in V(G)\), which is defined as the set of all vertices adjacent to \(v\) including \(v\) itself. By extension, the neighborhood \(N(S)\) of a set \(S \subseteq V(G)\) of vertices is defined as the union of the neighborhoods of the vertices \(v \in S\), i.e.,

\[
N(S) = \bigcup_{v \in S} N(v).
\]

Problem 5. Show that for every connected graph \(G\) the following inequalities hold:

\[
\text{rad}(G) \leq \text{diam}(G) \leq 2 \cdot \text{rad}(G).
\]

Find a graph where \(\text{rad}(G) = \text{diam}(G)\), and a graph where \(2 \cdot \text{rad}(G) = \text{diam}(G)\).

Problem 6. If the maximum degree of a connected bipartite graph \(G\) is \(\Delta(G)\), prove that the maximum number of vertices in it is

\[
|V(G)| \leq 2 \frac{(\Delta(G) - 1) \text{diam}(G) - 1}{\Delta(G) - 2}.
\]