Problem 1. Suppose that \( n \) people are attending a party and there are some handshakes between different people in the party. Show that there are at least two persons who have shaken hands with the same number of people.

*Hint. Pigeonhole.*

Problem 2. Balls of 8 different colors are placed in 6 jars. There are 20 balls of each color. Show that there must be a jar containing two pairs from two different colors of balls (for example, there is a jar containing at least two blue and at least two green balls).

Problem 3. Prove that for any \( n \in \mathbb{N} \),

\[
\sum_{i=0}^{n} \binom{n}{i} = 2^{n-1},
\]

where \( \binom{n}{i} = \frac{n!}{i!(n-i)!} \) denotes the binomial coefficient.

*Hint. There is a lot of ways in which one can prove the claim. One way is to show first that \( \sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0 \) with the binomial theorem.*

Problem 4. Let \( d = (d_1, d_2, \ldots, d_n) \) be a nonincreasing sequence of non-negative integers. We say that \( d \) is *graphic* if there exists a simple graph with degree sequence \( d \). Recall that in class we discussed that for the sequence \( d \) to be graphic we need that

\[
\sum_{i=1}^{n} d_i \text{ is even,}
\]

\[
\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min(k, d_i), \quad 1 \leq k \leq n.
\]

The aim of this exercise is to show an algorithm to construct a graph with degree sequence \( d \), if such a graph exists.

1. Suppose that \( d \) is graphic, i.e., there exists a graph \( G \) which has degree sequence \( d \). Show that there exists a graph \( G' \) s.t. the vertex with degree \( d_1 \) is connected to the vertices with degrees \( d_2, d_3, \ldots, d_{d_1+1} \).

2. Let \( d' = (d_2 - 1, d_3 - 1, \ldots, d_{1+d_1} - 1, d_{2+d_1}, \ldots, d_n) \). Use the previous result to show that \( d \) is graphic if and only if \( d' \) is graphic.

3. Now, what could be an algorithm which constructs a graph with degree sequence \( d \) (if such a graph exists)?

Note that with the procedure outlined in this exercise we can prove that the conditions (1) are also *sufficient* for the sequence \( d \) to be graphic.
For the next two problem we are going to need some definitions. Let \( d(u, v) \) denotes the graph distance between the vertices \( u \) and \( v \) \((u, v \in V(G))\), which is the minimum length of the paths connecting them, i.e., the length of the shortest path.

Let \( \text{diam}(G) \) denote the diameter of \( G \), which is the longest shortest path between any two vertices of the graphs, i.e.,
\[
\text{diam}(G) = \max_{u, v \in V(G)} d(u, v).
\]

Let \( \text{ecc}(v) \) denote the eccentricity of the vertex \( v \in V(G) \), which is defined as the maximum graph distance between \( v \) and any other vertex \( u \in V(G) \), i.e.,
\[
\text{ecc}(v) = \max_{u \in V(G)} d(u, v).
\]

Let \( \text{rad}(G) \) denote the radius of the graph \( G \), which is defined as the minimum graph eccentricity of any graph vertex in \( G \), i.e.,
\[
\text{rad}(G) = \min_{v \in V(G)} \text{ecc}(v).
\]

Let \( N(v) \) denote the neighborhood of a vertex \( v \in V(G) \), which is defined as the set of all vertices adjacent to \( v \) including \( v \) itself. By extension, the neighborhood \( N(S) \) of a set \( S \subseteq V(G) \) of vertices is defined as the union of the neighborhoods of the vertices \( v \in S \), i.e.,
\[
N(S) = \bigcup_{v \in S} N(v).
\]

**Problem 5.** Show that for every connected graph \( G \) the following inequalities hold:
\[
\text{rad}(G) \leq \text{diam}(G) \leq 2 \cdot \text{rad}(G).
\]

Find a graph where \( \text{rad}(G) = \text{diam}(G) \), and a graph where \( 2\text{rad}(G) = \text{diam}(G) \).

**Problem 6.** If the maximum degree of a connected bipartite graph \( G \) is \( \Delta(G) \), prove that the maximum number of vertices in it is
\[
|V(G)| \leq 2\left(\frac{\Delta(G)}{\Delta(G) - 2}\right)^{\frac{\text{diam}(G)}{\Delta(G)}} - 1.
\]