Problem 1. Given a permutation \( \sigma \) on \( \{1, 2, \ldots, n\} \), we say that \( i \in \{1, 2, \ldots , n\} \) is a fixed point of the permutation if \( \sigma(i) = i \). Pick a permutation uniformly at random from the set of all permutations on \( \{1, 2, \ldots , n\} \). What is the expected number of fixed points?

Problem 2. Take a circle, color 5/6 of it black and the rest red (not necessarily a contiguous part). We say that an inscribed regular pentagon is black if all its vertices are on the black part of the circle. Prove that there exists a black pentagon.

Hint. Pick an inscribed regular pentagon at random. What is the probability that it is black?

Problem 3. Recall that in Problem 4 of Problem Set 2 we proved that every tournament has a Hamiltonian path. The aim of this exercise is to show that there exists a tournament with a lot of Hamiltonian paths.

Show that there is a tournament on \( n \) vertices that has at least \( \frac{n!}{2^{n-1}} \) Hamiltonian paths.

Hint. Consider a random tournament \( T \) where the orientations of the edges are chosen independently and uniformly. Write the expected number of Hamiltonian paths as the sum of expectations of indicator functions.

Problem 4. Let \( G(n, p) \) be a graph with \( n \) vertices s.t. for any pair of nodes the edge connecting them is included in the graph with probability \( p \) independently from every other edge. In this exercise, we are interested in finding how to choose \( p \) as a function of \( n \) (for \( n \) large) so that with high probability \( G(n, p) \) contains a triangle (clique of size 3).

a) Suppose that \( p \) decays faster than \( 1/n \), i.e. \( p(n) = o(1/n) \). Show that the probability that \( G(n, p) \) contains a triangle tends to 0.

b) Suppose that \( p \) decays slower than \( 1/n \), i.e. \( 1/n = o(p(n)) \). Show that the probability that \( G(n, p) \) contains a triangle tends to 1.

Note. If conditions a) and b) hold, we say that \( 1/n \) is a threshold function for the property “containing a triangle”.