Graded Problem Set 12

Date: 6.12.2013 Due date: 13.12.2013, noon

Instructions: Please write the solution for each problem on a separate piece of paper. Sort these pieces of paper, with Problem 1 on top. Add this page as cover page, fill in your name, Sciper number, and the list of collaborators, and staple them all together on the left top.

Rules: You are allowed and encouraged to discuss these problems with your colleagues. However, each of you has to write down her solution in her own words. If you collaborated on a homework, write down the name of your collaborators and your sources in the space below. No points will be deducted for collaborations. But if we find similarities in solutions beyond the listed collaborations we will consider it as cheating. Please note that EPFL has a VERY strict policy on cheating and you might be in BIG trouble. It is simply not worth it.

Grading: Each problem is worth 20 points. 10 additional points can be gained solving Problem 5 part b).

Collaborators and sources:

First name: ....

Last name: ....

Sciper: ....

Problem 1 ... / 20

Problem 2 ... / 20

Problem 3 ... / 20

Problem 4 ... / 20

Problem 5 ... / 30

TOTAL ... / 100

Problem 1.

a) Consider the sequence whose generating function is $F(x) = \prod_{k=1}^{5} \frac{1}{(k-x)^k}$. Find a Θ-approximation for this sequence (of the form n^{42}, $n3^n$, n^{-3}, $\log n$, etc).

b) Do the same for $G(x) = xF(x)$.

c) Do the same for $H(x) = F(x) - G(x)$.

Problem 2. You need to pay n dollars to buy a hamburger. In how many ways can you do this using 1\$ coins, 1\$ bills and 2\$ bills (just like last week), where (unlike last week) the order does not matter? For example, you can pay 2\$ in 4 ways: two 1\$ coins; two 1\$ bills; one 1\$ coin and one 1\$ bill; one 2\$ bill. The use of generating functions is recommended!

Problem 3. You throw a fair coin $2n$ times, where $n \in \mathbb{N}_{\geq 1}$.

i) What is the probability of getting $2n$ Heads?

ii) What is the probability of getting 2 Tails?

iii) Suppose that you want to bet with a friend on the number of Tails that you are going to observe. On what number should you bet?

iv) Suppose now that the coin is biased s.t. the probability of getting Tails in a single trial is $\frac{1}{3}$. The $2n$ trials remain independent and equiprobable. Compute again the probabilities of point i) and ii).

Problem 4. Urn 1 contains 2 blue tokens and 8 red tokens; urn 2 contains 12 blue tokens and 3 red tokens. You roll a die to determine which urn to choose: if you roll a 1 or 2 you choose urn 1; if you roll a 3, 4, 5, or 6 you choose urn 2. Once the urn is chosen, you draw out a token at random from that urn. Given that the token is blue, what is the probability that the token came from urn 1?

Problem 5. If we sum the outcomes of 2 dice with 6 equiprobable faces, we obtain the integers from 2 to 12 with different probabilities. The aim of this problem is to show that it is impossible to modify the probabilities of obtaining each face of each die so that the sum of the outcomes of the dice is a uniform random variable with values in $\{2, 3, \cdots, 12\}$.

Suppose by contradiction that the claim is false. Let p_i and q_i be the probabilities that the face i is obtained in the first and the second die, respectively ($i \in \{1, 2, \cdots, 6\}$).

a1) As a function of p_i and q_i ($i \in \{1, 2, \cdots, 6\}$), what is the probability that the sum of the outcomes is 2? Since the sum of the outcomes of the dice is uniform in $\{2, 3, \cdots, 12\}$, what condition do we obtain on the p_i and the q_i?

a2) Same question for the sum of the outcomes equal to 12.

a3) Let $a, b \in \mathbb{R}_{\geq 0}$. Prove that
$$\frac{a+b}{2} \geq \sqrt{ab}.$$

a4) What is the probability that the sum of the outcomes of the two dice is 7? Bound this result with a function of p_1, q_1, p_6, and q_6. Then, using the previous points, find a contradiction and prove the claim.

b) (*Bonus*) The aim of this part is to show an alternative proof of the claim which uses generating functions. Let $p(x) = \sum_{i=1}^{6} p_i x^i$ and $q(x) = \sum_{i=1}^{6} q_i x^i$ be the generating functions associated to the probability distributions of the first and the second die, respectively.

What is the generating function $s(x)$ of the sum of the two dice? How can you write it using the fact that the sum of the outcomes is a uniform random variable? From these consideration, deduce a contradiction.

Hint. Recall that a polynomial of odd degree has always at least 1 real root.