Problem 1. A vending machine dispensing books of stamps accepts only $1 coins, $1 bills and $2 bills. Let $a_n$ denote the number of ways of depositing $n$ dollars in the vending machine, where the order in which the coins and bills are deposited matters.

(a) Find a recurrence relation for $a_n$ and give the necessary initial condition(s).

(b) Find an explicit formula for $a_n$ by solving the recurrence relation in part (a).

Problem 2. Solve the following recurrence relations using the characteristic equation.

i) $a_n = 5a_{n-1} - 4a_{n-2}$, with $a_0 = 1$ and $a_1 = 0$.

ii) $a_n = 5a_{n-1} - 4a_{n-2}$, with $a_0 = 0$ and $a_1 = 1$.

iii) $a_n = -10a_{n-1} - 21a_{n-2}$, with $a_0 = 2$ and $a_1 = 1$.

iv) $a_n = a_{n-2}$, with $a_0 = 2$ and $a_1 = -1$.

v) $a_n = 2a_{n-1} + 2a_{n-2}$, with $a_0 = 0$ and $a_1 = 1$.

vi) $a_n = 2a_{n-1} - a_{n-2}$, with $a_0 = 3$ and $a_1 = 5$.

vii) $a_n = -6a_{n-1} - 11a_{n-2} - 6a_{n-3}$, with $a_0 = 0$, $a_1 = 1$, and $a_2 = 2$.

viii) $a_n = 10a_{n-1} - 37a_{n-2} + 60a_{n-3} - 36a_{n-4}$, with $a_0 = 0$, $a_1 = 0$, $a_2 = 1$, and $a_3 = 0$.

Problem 3. Solve the same recurrence relations of Problem 2 using generating functions.

Problem 4. Find the coefficient of $x^8$ in the power series of each of the following functions:

(a) $\frac{1}{1 - 2x}$

(b) $\frac{x^3}{1 - 3x}$

(c) $\frac{1}{(1 - x)^2}$

(d) $\frac{x^2}{(1 + 2x)^2}$

(e) $\frac{1}{1 - 3x^2}$

(f) $x^3 \cdot \frac{5 + 2x - 21x^2}{2x^3 - x^2 - 2x + 1}$