ECOLE POLYTECHNIQUE FEDERALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 13 Statistical Physics for Communication and Computer Science
Final Project - due date: July 2nd 2013 May 22, 2013

In the course we discussed in some detail the AMP algorithm in the framework of the Lasso estimator. The aim of this project is to explore in more detail other aspects that were discussed in less detail. You will consider AMP in a framework were the prior is known, explore the phase diagram, compare to the Lasso phase diagram, and also implement the idea of spatial coupling for this problem.

The formalism that you will need was developed in chapters 12 and 17 of the notes so we will not repeat all the equations here. The items below can be solved mostly numerically but when possible try to first simplify analytically your calculations. We expect you to try a few reasonable values of parameters in an explorative manner. We also expect you to hand us back a short report explaining your results and illustrating them with plots.

You will explore the case of the Bernoulli-Gaussian prior:

$$p_0(x) = (1-\epsilon)\delta(x) + \epsilon\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \tag{1}$$

(A) *Fixed point equation in the Lasso case.*
This is a preliminary question. In fact you can skip it; it is independent from the other ones. Verify that the fixed point equation of state evolution for the Lasso estimator has at most a unique fixed point. Analyze the fixed point equation graphically and numerically.

(B) *Thresholding function for known priors.*
Now, consider the framework of Chapter 17 where the prior is known. Compute analytically the thesholding function $\eta_0(y,\nu)$. Compute for later use (numerically) $\tilde{\delta}(p_0) = \sup_\nu \nu^{-2}\mathrm{mmse}(\nu^{-2})$ as a function of ϵ. Also for later use, verify the inequality $\tilde{\delta}(p_0) > \epsilon$ (this can be done analytically).

(C) *Fixed point equation for known priors.*
The goal of this question is analyze the fixed point equation of state evolution with known prior (1). We ask you to verify the following points:

- for $\delta > \tilde{\delta}(p_0)$ there is a unique fixed point $O(\sigma^2)$.
- for $\tilde{\delta}(p_0) < \delta < \epsilon$ there are at least two stable fixed points: one is $O(\sigma^2)$, and the other one $\Theta(1)$.
- What is the natural initial condition for the state evolution ? Show that under this initial condition iterations always tend to the fixed point $\Theta(1)$ (for δ in the range $[\tilde{\delta}(p_0), \epsilon]$).

(D) *Phase diagram.*
Use the above results to plot on the same (ϵ, δ)-plane the optimal, Lasso and AMP with known prior phase transition lines. Note that in Chapter 12 we gave the phase transition line of Lasso in another system of variables.

(D) *Comparison of MSE's of AMP and state evolution.*
Run AMP for about 30 instances with $\sigma \approx 0.01$ and $\epsilon \approx 0.1$. Plot the average MSE as a function of iterations (approx 1000 iterations) and compare to the MSE obtained through state evolution. It is better to use a logarithmic scale for the MSE.

(E) *Spatial coupling.* Consider the spatially coupled setting. The goal of this question is to observe the "estimation wave". Calculate numerically the solution of one-dimensional state evolution equation. Try values around $\sigma \approx 0.01$, $\epsilon \approx 0.1$, $L = 500$, $N = 50$, $M = 6$. So the signal has 25000 components and the sampling rate (in the bulk of the chain) is $\delta \approx 0.12$. Try various $w \geq 3$, try to take the variances of the sensing matrix given in the notes and try to play around with these variances in the upper-left corner (of size $O(w) \times (w)$) to improve the seeding mechanism.