ECOLE POLYTECHNIQUE FEDERALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 12 Statistical Physics for Communication and Computer Science
Homework 12 May 16, 2013

Problem 1 (Magnetization of the Ising model on a d-regular graph with large girth)**.** In this problem we consider the ferromagnetic Ising model on a d-regular graph with large girth. Using the probabilistic method Erdős and Sachs proved that there exist a graphs $G_{n,d}$ on n vertices, with all vertex degrees equal to d and with a girth $g_{n,d} \geq (1 - o(1)) \log_{d-1} n$ (here $o(1)$ stands for a function that goes to zero as $n \to +\infty$). We recall that the girth is the length of the shortest loop in the graph.

Consider the Gibbs distribution of the Ising model on $G_{n,d}$

$$\mu_{n,d}(\underline{s}) = \frac{1}{Z_{n,d}} \exp\biggl(\frac{\beta J}{d} \sum_{\{i,j\} \in \text{edges}} s_i s_j + \beta h \sum_{i=1}^{n} s_i\biggr)$$

The Hamiltonian is given by the contribution of all ferromagnetic interactions associated to edges $\{i, j\}$, and a contribution from a constant magnetic field. The strength of the interaction is scaled by d for later convenience. Note that $J > 0$ but h can take both signs.

Recall that the magnetization at a vertex o is defined as $\langle s_o \rangle_{n,d}$ where $\langle - \rangle_{n,d}$ is the usual Gibbs average. This quantity is non trivial to compute. On the other hand we can run BP and compute the BP estimates of the magnetization.

(i) The second Griffith-Kelly-Sherman correlation inequality states that for Ising models with all interaction coefficients and all magnetic fields positive the magnetization can only decrease when one coefficient decreases. In the present case this inequality implies that the magnetization decreases when an edge is removed from $G_{n,d}$. Now consider the neighborhood of a vertex o, namely $N = \{i \in G_{n,d} | \text{dist}(o, i) \leq g_{n,d} - 1\}$. Define $\langle - \rangle_N$ the Gibbs average for the Ising model restricted to N. Show that for $h \geq 0$

$$\langle s_o \rangle_{n,d} \geq \langle s_o \rangle_N$$

and that for $h \leq 0$

$$\langle s_o \rangle_{n,d} \leq \langle s_o \rangle_N$$

Hint: for the second inequality use symmetry properties under the operation $h \to -h$.

(ii) The average $\langle s_o \rangle_N$ can be computed exactly from the BP recursion. Why? Show that this recursion is:

$$m^{(t)} = \tanh(\beta h + d \tanh^{-1}(\tanh \beta \frac{J}{d} \tanh u^{(t)}))$$

$$u^{(t)} = \beta h + (d - 1) \tanh^{-1}(\tanh \frac{\beta J}{d} \tanh u^{(t-1)}), \qquad u^{(0)} = h$$

and that $\langle s_o \rangle_N = m^{(g_{n,d}-1)}$.

Remark: go back to homework 4 and observe this is the same recursion that you had derived by "other means".

(iii) Take now a fixed sequence of graphs $G_{n,d}$ with respect to n. Observe from above that for $h > 0$ and all t,

$$\liminf_{n\to+\infty} \langle s_o \rangle_{n,d} \geq m^{(t)},$$

and for $h \geq 0$

$$\limsup_{n\to+\infty} \langle s_o \rangle_{n,d} \leq m^{(t)}.$$

We want to look at the limit $d \to +\infty$. Show that

$$\lim_{d\to+\infty} \liminf_{n\to+\infty} \langle s_o \rangle_{n,d} \geq \lim_{t\to+\infty} m_{\mathrm{CW}}^{(t)},$$

and for $h \leq 0$ and all t

$$\lim_{d\to+\infty} \limsup_{n\to+\infty} \langle s_o \rangle_{n,d} \leq \lim_{t\to+\infty} m_{\mathrm{CW}}^{(t)},$$

where $m_{\mathrm{CW}}^{(t)}$ is the BP-magnetization of the CW model and satisfies the recursion

$$m_{\mathrm{CW}}^{(t)} = \tanh(\beta(h + J m_{\mathrm{CW}}^{(t-1)}))$$

with the initial condition $m_{\mathrm{CW}}^{(o)} = \tanh \beta h$.

Remark: These inequalities suggest the conjecture

$$\lim_{d\to+\infty} \liminf_{n\to+\infty} \langle s_o \rangle_{n,d} = \lim_{d\to+\infty} \limsup_{n\to+\infty} \langle s_o \rangle_{n,d} = \langle s_o \rangle_{\mathrm{CW}}$$

where $\langle s_o \rangle_{\mathrm{CW}}$ is the true CW magnetization.