# Polarization and polar codes

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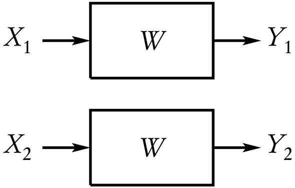
Arıkan's polar coding is a technique to convert any binary-input channel to a mixture of binary-input extremal channels.

- The technique is information lossless, and of low complexity$^{\star}$.
- I am not the inventor of this technique.

# Building block

Given two copies of a binary input channel $W : \mathbb{F}_2 \to \mathscr{Y}$



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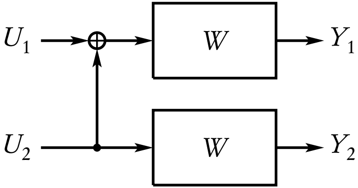
$$X_1 = U_1 \oplus U_2$$
$$X_2 = U_2$$

with $U_1, U_2$ i.i.d., uniform on $\mathbb{F}_2$.



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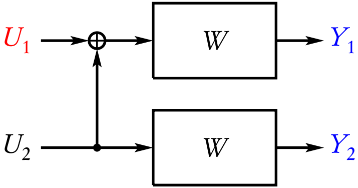


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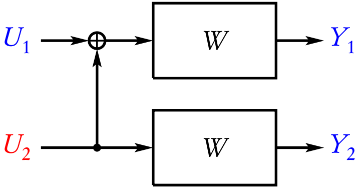


- This induces two synthetic channels $W^- : \mathbb{F}_2 \to \mathscr{Y}^2$ and $W^+ : \mathbb{F}_2 \to \mathscr{Y}^2 \times \mathbb{F}_2$.

# Building block

Note that

$$W^{-}(y_1, y_2 | u_1) = \sum_{u_2 \in \mathbb{F}_2} \tfrac{1}{2} W(y_1 | u_1 \oplus u_2) W(y_2 | u_2)$$

$$W^{+}(y_1, y_2, u_1 | u_2) = \tfrac{1}{2} W(y_1 | u_1 \oplus u_2) W(y_2 | u_2)$$

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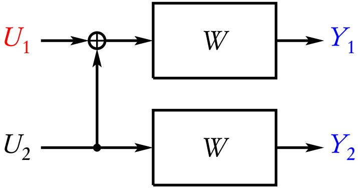
Properties of $W \mapsto (W^-, W^+)$:

$$I(W^-) = I(U_1; Y_1 Y_2)$$



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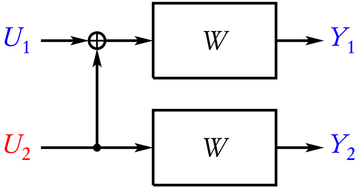

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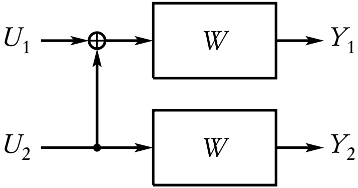

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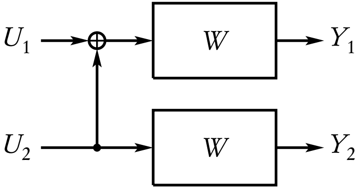

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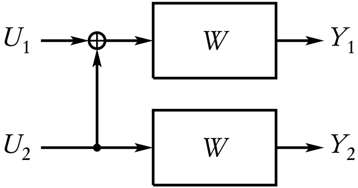


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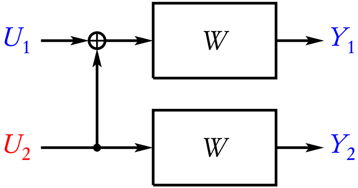


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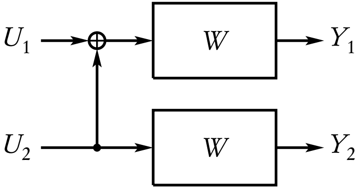


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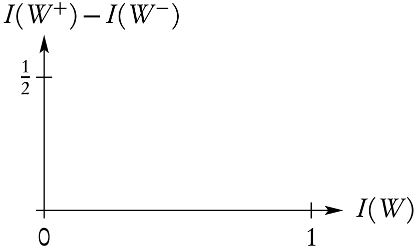

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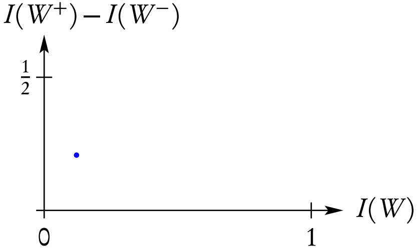

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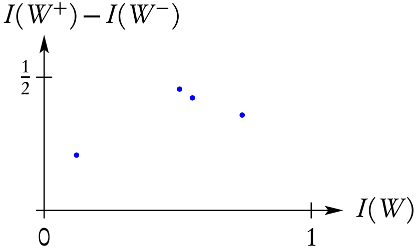

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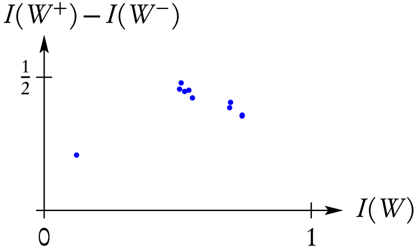

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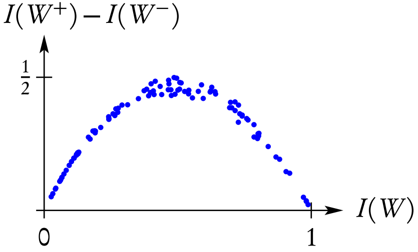

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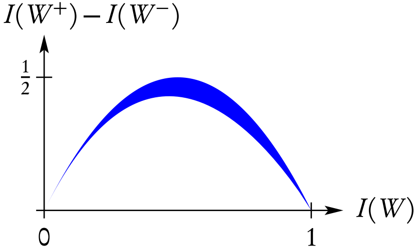

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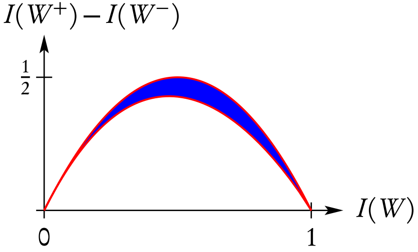

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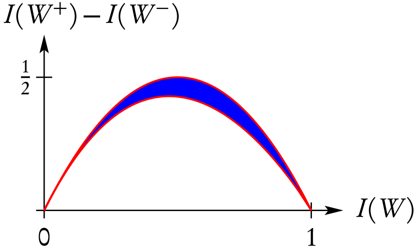
- $I(W^+) \geq I(W) \geq I(W^-)$.
- For every $\epsilon > 0$ there is a $\delta > 0$ such that

  $$I(W^+) - I(W^-) < \delta \text{ implies}$$

  $$I(W) \notin (\epsilon, 1 - \epsilon).$$



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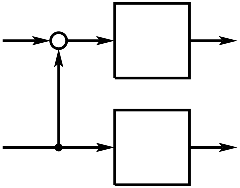

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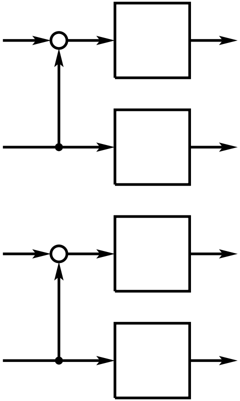

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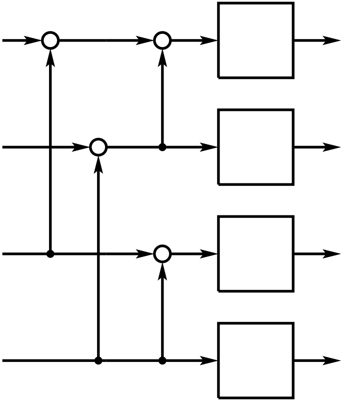

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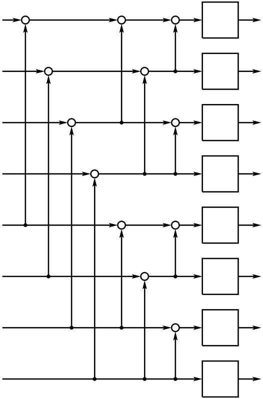

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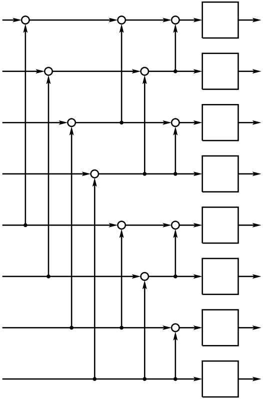

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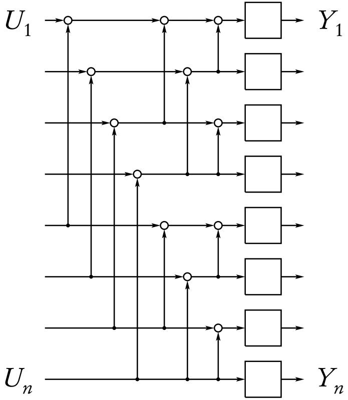
- $\ell$ levels into this process, we have transformed $n = 2^\ell$ uses of channel $W$ to one use each of $2^\ell$ channels

$$W^{b_1 \dots, b_\ell}, \quad b_j \in \{+, -\},$$

these are the channels

$$U_i \to Y^n U^{i-1}, \quad i = 1, \dots, n.$$



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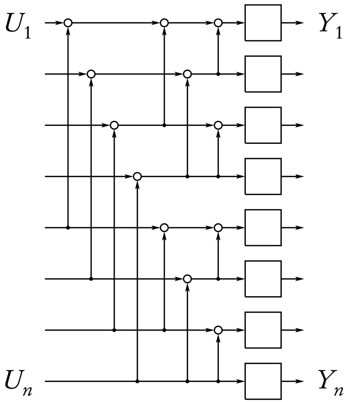
- The quantities $\{I(W^{b_1\ldots b_\ell})\}$ are exactly the $n$ quantities

$$I(U_i; Y^n U^{i-1}), \quad i = 1,\ldots,n$$



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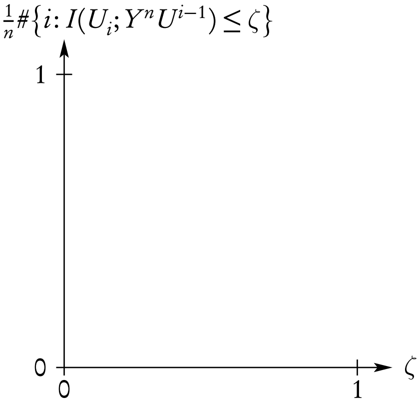
$W$ is a binary erasure channel, $p = 0.4$



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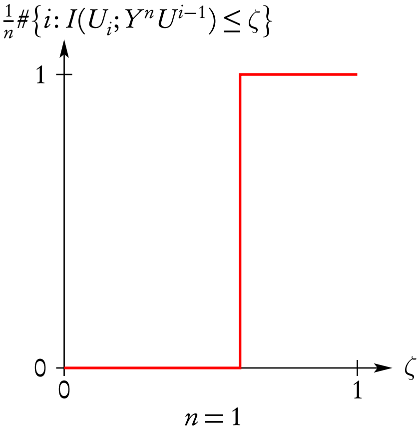

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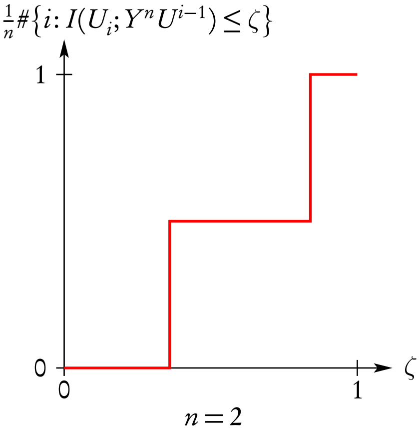

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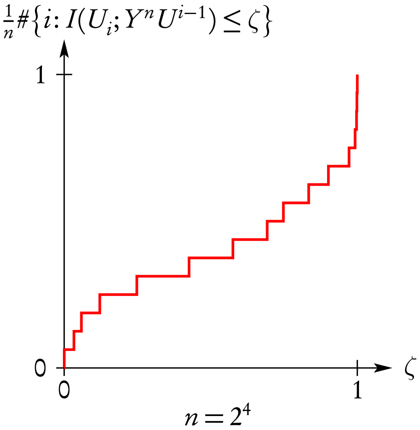

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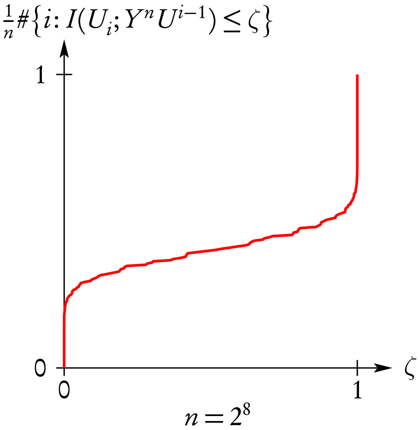

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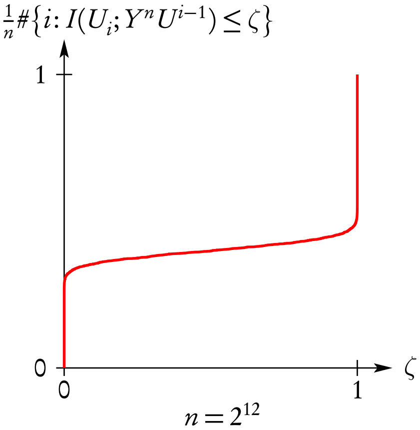

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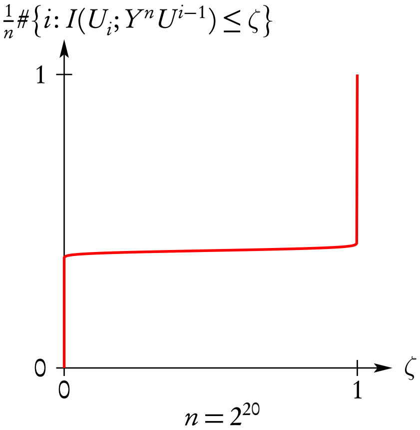

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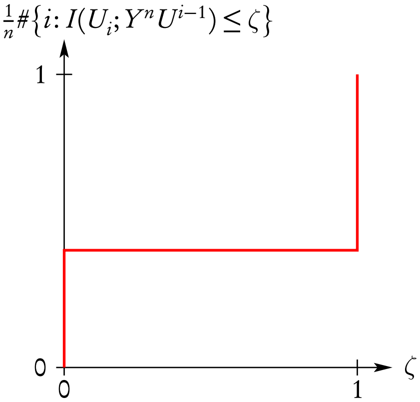

# Polarization

- Polarization refers to the phenomenon that in the limit, almost all channels are extremal, i.e.,

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- The decoder can decode $U_1, U_2, \ldots, U_n$ successively.

# On successive decoding

The quantities $I(U_i; Y^n U^{i-1})$ are relevant for a genie-aided decoder:

$$\hat{U}_1 = \phi_1(Y^n)$$
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$$\hat{U}_n = \phi_n(Y^n, \hat{U}^{n-1}).$$

If the genie-aided decoder makes no errors, then, the unaided decoder makes no errors.

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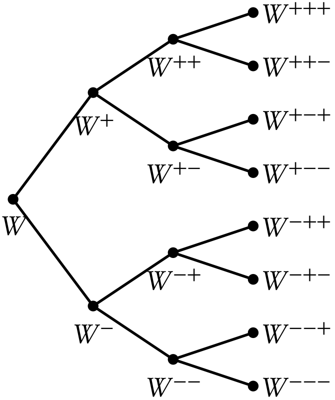

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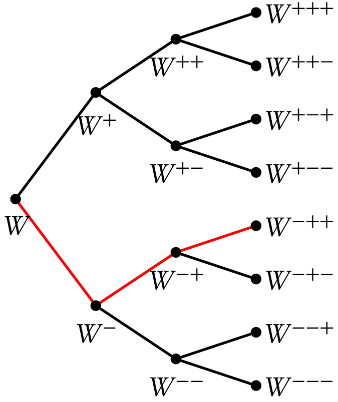

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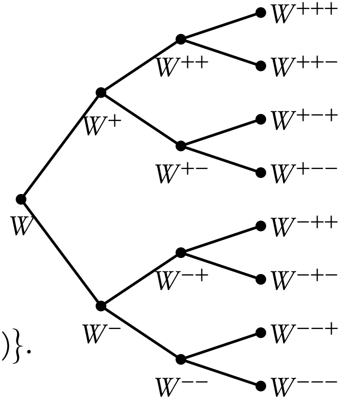
- $W_\ell$ is uniformly distributed among $\{W^{-\cdots-}, \dots, W^{+\cdots+}\}$, and

$$\Pr\big(I(W_\ell) \in (\epsilon, 1-\epsilon)\big) = \frac{1}{n}\#\big\{i \colon I(U_i; Y^n U^{i-1}) \in (\epsilon, 1-\epsilon)\big\}.$$



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- so,

$$E[I_{\ell+1}|B_1,\ldots,B_\ell] = \tfrac{1}{2}[I(W_\ell^-)+I(W_\ell^+)] = I(W_\ell) = I_\ell$$

and we see that $\{I_\ell\}$ is a martingale.

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- But how fast? Fast enough to arrest error propagation?
- Introduce the Bhattacharyya parameter

$$Z(W) = \sum_y \sqrt{W(y|0)W(y|1)}$$

as a companion to $I(W)$. Note that this is an upper bound on probability of error for uncoded transmission over $W$.

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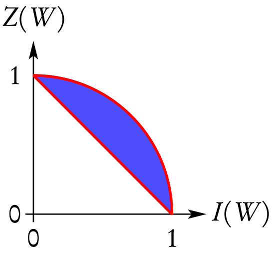

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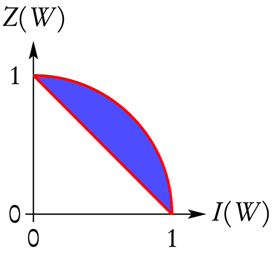

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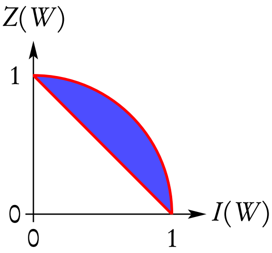

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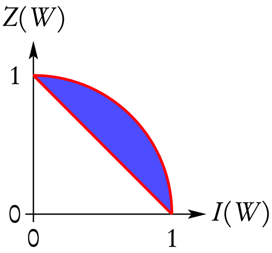

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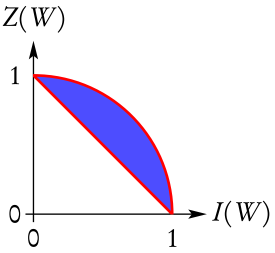


Since $Z(W)$ upper bounds on probability of error for uncoded transmission over $W$, we can choose the good indices on the basis of the $Z$'s of the synthetic channels. The sum of the $Z$'s of the chosen channels will upper bound the block error probability. This suggests studying the polarization speed of $Z$.

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- This means that for any $\beta < 1/2$, as long as $R < I(W)$ the error probability of polarization codes decays to 0 faster than $2^{-n^\beta}$.

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- With '$k$-by-$k$' constructions the error probability can be made to decay almost exponentially in blocklength.

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- For channel coding applications, the usual suspects (LDPC, Turbo, ...) easily beat the pure polar codes at meaningful block lengths. For quantization they are a lot more competitive.

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- The technique treats noise not by eliminating it, but by shifting it to a subspace.
- Successive decoding is cheap ($n \log n$) but too naive. More clever decoding methods should improve error probability at moderate $n$.
- While the original motivation was for channel coding, polar codes are good at many settings. [“Polar codes are good for everything — S. Korada”]