ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 14 — Information Theory and Coding

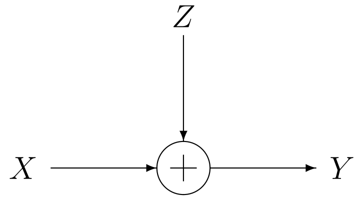
Homework 7 — October 30, 2012

Problem 1. One is given a communication channel with transition probabilities $p(y|x)$ and channel capacity $C = \max_{P_X} I(X;Y)$. A helpful statistician preprocesses the output by forming $\tilde{Y} = g(Y)$. He claims that this will strictly improve the capacity.

(a) Show that he is wrong.

(b) Under what conditions does he not strictly decrease the capacity?

Problem 2. Find the channel capacity of the following discrete memoryless channel:



where $\Pr\{Z = 0\} = \Pr\{Z = a\} = \frac{1}{2}$ and $a \neq 0$. The alphabet for x is $\mathcal{X} = \{0, 1\}$. Assume that Z is independent of X.

Observe that the channel capacity depends on the value of a.

Problem 3. Consider the discrete memoryless channel $Y = X + Z \pmod{11}$, where

$$\Pr(Z = 1) = \Pr(Z = 2) = \Pr(Z = 3) = 1/3$$

and $X \in \{0, 1, \dots, 10\}$. Assume that Z is independent of X.

(a) Find the capacity.

(b) What is the maximizing $p^*(x)$?

Problem 4. The Z-channel has binary input and output alphabets and transition probabilities $p(y|x)$ given by

$$p(0|0) = 1 \quad \text{and} \quad p(0|1) = \varepsilon.$$

Find the capacity of the Z-channel and the maximizing input probability distribution in terms of ε.

Problem 5. Consider two discrete memoryless channels. The first channel has input alphabet $\mathcal{X}$, output alphabet $\mathcal{Y}$; the second channel has input alphabet $\mathcal{Y}$ and output alphabet $\mathcal{Z}$. The first channel is described by the conditional probabilities $P_1(y|x)$ and the second channel by $P_2(z|y)$. Let the capacities of these channels be C_1 and C_2. Consider a third memoryless channel described by probabilities

$$P_3(z|x) = \sum_{y \in \mathcal{Y}} P_2(z|y) P_1(y|x), \quad x \in \mathcal{X},\ z \in \mathcal{Z}.$$

Show that the capacity C_3 of this third channel satisfies

$$C_3 \leq \min\{C_1, C_2\}.$$

Problem 6. Suppose we have two channels $(\mathcal{X}_i, p_i(y_i|x_i), \mathcal{Y}_i)$, $i = 1, 2$ with capacities C_i, $i = 1, 2$ and let assume that $\mathcal{X}_1 \cap \mathcal{X}_2 = \emptyset$ and $\mathcal{Y}_1 \cap \mathcal{Y}_2 = \emptyset$. We are going to communicate by using these channels but we are not allowed to use both of them. Hence, every time we want to send a message, we select one of the channels and as $\mathcal{Y}_1 \cap \mathcal{Y}_2 = \emptyset$ the receiver will automatically know which channel we have used. There are different schemes for communication, for example, we can always use the channel with maximum capacity to transfer $\max\{C_1, C_2\}$ bits in one access to the channel but we want to show that we can do better!!

1. Convince yourself that in this case the communication channel is simply $(\mathcal{X}_1 \cup \mathcal{X}_2, p_{12}(y|x), \mathcal{Y}_1 \cup \mathcal{Y}_2)$, where

$$p_{12}(y|x) = \begin{cases} p_1(y|x) & x \in \mathcal{X}_1, y \in \mathcal{Y}_1, \\ p_2(y|x) & x \in \mathcal{X}_2, y \in \mathcal{Y}_2, \\ 0 & \text{otherwise.} \end{cases}$$

2. Assume that the capacity achieving distributions for the channels are $p_1^\star$ and $p_2^\star$ respectively. Show that the capacity achieving distribution for the combined channel is of the form

$$p^\star(x) = \begin{cases} \alpha p_1(x) & x \in \mathcal{X}_1, \\ (1-\alpha) p_2(x) & x \in \mathcal{X}_2, \end{cases}$$

for some $0 \leq \alpha \leq 1$.

3. Find the mutual information of the combined channel as a function of α and by optimizing it over α prove that the following relation holds

$$2^C = 2^{C_1} + 2^{C_2},$$

where C is the capacity of the combined channel.

4. Assume that $C_1 = C_2 = 0$. What is the resulting capacity? How can this result be true when we know that no information can be transmitted directly through individual channels?