ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 4 Information Theory and Coding
Homework 2 September 25, 2012

PROBLEM 1. Let $\bar{M} = \sum_{i=1} p_i l_i^{100}$ be the 100th moment (i.e., the expected value of the 100th power) of the code word lengths l_i associated with an encoding of a random variable X with distribution p. Let $\bar{M}_1 = \min \bar{M}$ over all prefix-free codes for X; and let $\bar{M}_2 = \min \bar{M}$ over all uniquely decodable codes for X. What relationship exists between $\bar{M}_1$ and $\bar{M}_2$?

PROBLEM 2. During an oral exam, a Mischievious Teaching Assistant (MTA) prepares 2 very hard questions, and 1 very easy question and puts them in 3 identical boxes. The student is then allowed to pick one box, without looking inside. In order to maximize his chances of getting a perfect grade, the student would like to pick the box containing the easy question.

(a) What is the probability of picking the very easy question?

The MTA then announces to the student that he will remove one of the hard questions. He opens one of the two remaining boxes, and proceeds to read the question to the student, confirming it was indeed a very hard question. The MTA then offers the student the choice to exchange the box he picked at the beginning for the remaining, unpicked box.

(b) Would you keep the original box or exchange it? What is the probability that the very easy question is in the unpicked box?

PROBLEM 3. Consider the following method for constructing binary code words for a random variable U which takes values $\{a_1, \dots, a_m\}$ with probabilities $P(a_1), \dots, P(a_m)$. Assume that $P(a_1) \geq P(a_2) \geq \cdots \geq P(a_m)$. Define

$$Q_1 = 0 \quad \text{and} Q_i = \sum_{k=1}^{i-1} P(a_k) \quad \text{for } i = 2, 3, \dots$$

The code word assigned to the letter a_i is formed by finding the binary expansion of $Q_i < 1$ (i.e, $1/2 = .100\dots$, $1/4 = .0100\dots$, $5/8 = .1010\dots$) and letting the codeword be the first l_i bits of this expansion where $l_i = \lceil -\log_2 P(a_i) \rceil$.

(a) Construct binary code words for the probability distribution $\{1/4, 1/4, 1/8, 1/8, 1/16, 1/16, 1/16, 1/16\}$.

(b) Prove that the method described above yields a prefix-free code and the average codeword length $\bar{L}$ satisfies

$$H(X) \leq \bar{L} < H(X) + 1.$$

PROBLEM 4. A random variable takes values on an alphabet of K letters, and each letter has the same probability. These letters are encoded into binary words using the Huffman procedure so as to minimize the average code word length. Let j and x be chosen such that $K = x2^j$, where j is an integer and $1 \leq x < 2$.

(a) Do any code words have lengths not equal to j or $j+1$? Why?

(b) In terms of j and x, how many code words have length j?

(c) What is the average code word length?

PROBLEM 5. Let $p_{XY}(x, y)$ be given by

$X \backslash Y$	0	1
0	1/3	1/3
1	0	1/3

Find

(a) $H(X)$, $H(Y)$.

(b) $H(X|Y)$, $H(Y|X)$.

(c) $H(X, Y)$.

(d) $H(Y) - H(Y|X)$.

(e) $I(X; Y)$.

(f) Draw a Venn diagram for the quantities in (a) through (e).

PROBLEM 6. Let X be a random variable taking values in M points $a_1, \ldots, a_M$, and let $P_X(a_M) = \alpha$. Show that

$$H(X) = \alpha \log \frac{1}{\alpha} + (1 - \alpha) \log \frac{1}{1 - \alpha} + (1 - \alpha) H(Y)$$

where Y is a random variable taking values in $M-1$ points $a_1, \ldots, a_{M-1}$ with probabilities $P_Y(a_j) = P_X(a_j)/(1 - \alpha)$; $1 \leq j \leq M - 1$. Show that

$$H(X) \leq \alpha \log \frac{1}{\alpha} + (1 - \alpha) \log \frac{1}{1 - \alpha} + (1 - \alpha) \log(M - 1)$$

and determine the condition for equality.

PROBLEM 7. Let X, Y, Z be discrete random variables. Prove the validity of the following inequalities and find the conditions for equality:

(a) $I(X, Y; Z) \geq I(X; Z)$.

(b) $H(X, Y|Z) \geq H(X|Z)$.

(c) $H(X, Y, Z) - H(X, Y) \leq H(X, Z) - H(X)$.

(d) $I(X; Z|Y) \geq I(Z; Y|X) - I(Z; Y) + I(X; Z)$.