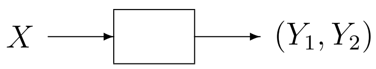
ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 20 Information Theory and Coding
Homework 10 November 27, 2012

PROBLEM 1.



Consider the ordinary Shannon Gaussian channel with two correlated looks at X, i.e., $Y = (Y_1, Y_2)$, where

$$\begin{aligned} Y_1 &= X + Z_1 \\ Y_2 &= X + Z_2 \end{aligned}$$

with a power constraint P on X, and (Z_1, Z_2) a Gaussian zero mean random vector with covariance K, where

$$K = \begin{bmatrix} N & N\rho \\ N\rho & N \end{bmatrix}.$$

Find the capacity C for

(a) $\rho = 1$.

(b) $\rho = 0$.

(c) $\rho = -1$.

PROBLEM 2. Consider a pair of parallel Gaussian channels, i.e.,

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix},$$

where

$$\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} \right),$$

and there is a power constraint $E(X_1^2 + X_2^2) \leq 2P$. Assume that $\sigma_1^2 > \sigma_2^2$.

(a) Suppose we use the capacity achieving distribution as input. At what power does the channel stop behaving like a single channel with noise variance σ_2^2, and begin behaving like a pair of channels?

(b) Let $C_1(P)$ be the capacity of the pair of gaussian channels when the input is constrained to have a power not exceeding $2P$. Let $C_2(P) = I(X_1, X_2; Y_1, Y_2)$ when both X_1 and X_2 are independent gaussian random variables with variance equal to P. Show that $C_1(P) - C_2(P)$ tends to zero as P/σ_1^2 tends to infinity.

PROBLEM 3. Consider a vector Gaussian channel described as follows:

$$Y_1 = x + Z_1$$
$$Y_2 = Z_2$$

where x is the input to the channel constrained in power to P; Z_1 and Z_2 are jointly Gaussian random variables with $E[Z_1] = E[Z_2] = 0$, $E[Z_1^2] = E[Z_2^2] = \sigma^2$ and $E[Z_1 Z_2] = \rho\sigma^2$, with $\rho \in [-1, 1]$, and independent of the channel input.

(a) Consider a receiver that discards Y_2 and decodes the message based only on Y_1. What rates are achievable with such a receiver?

(b) Consider a receiver that forms $Y = Y_1 - \rho Y_2$, and decodes the message based only on Y. What rates are achievable with such a receiver?

(c) Find the capacity of the channel and compare it to the part (b).

PROBLEM 4.

a) Let x^* be the most probable letter of a finite source $\mathcal{X}$, i.e. $P(x^*) \geq P(x)$, for all $x \in \mathcal{X}$. Show that
$$H(X) \geq \log(\frac{1}{P(x^*)}).$$

b) [Fano's Inequality] Assume that $\mathcal{X}$ generates a letter and we want to estimate the outcome of $\mathcal{X}$ by observing random variable Y which is related to X by the conditional distribution $p(y|x)$. From Y, we calculate a function $g(Y) = \hat{X}$, where $\hat{X}$ is an estimate of X. Let P_e be the error probability of estimation defined as $P_e = P\{\hat{X} \neq X\}$. Prove that
$$H(X \mid Y) \leq H(P_e) + P_e \log(|\mathcal{X}| - 1),$$
where $|\mathcal{X}|$ denotes the number of letters in the alphabet $\mathcal{X}$.

c) [Fano's Inverse Inequality] Assume that we use a *Maximum A Posteriori* estimator, i.e. for an observation y,
$$\hat{x} = g(y) = \arg\max_{x \in \mathcal{X}} p(x|y).$$
Prove that
$$P_e \leq 1 - 2^{-H(X|Y)}.$$
Hint: use part (a) and note that $\sum_i p_i 2^{-u_i} \geq 2^{-\sum_i p_i u_i}$.

PROBLEM 5. If $X_1, X_2, \ldots$ are random variables such that $E[X_i|X^{i-1}] = X_{i-1}$, show that $E[(X_{i+1} - X_i)(X_{j+1} - X_j)] = 0$ if $i \neq j$.

PROBLEM 6. Compute the transition probabilities of the synthesized + and − channels after applying the basic polarization transformations you have seen in class starting from a binary symmetric channel with crossover probability ϵ