Consider the ordinary Shannon Gaussian channel with two correlated looks at $X$, i.e., $Y = (Y_1, Y_2)$, where

$$
Y_1 = X + Z_1 \\
Y_2 = X + Z_2
$$

with a power constraint $P$ on $X$, and $(Z_1, Z_2)$ a Gaussian zero mean random vector with covariance $K$, where

$$
K = \begin{bmatrix} N & N\rho \\
N\rho & N \end{bmatrix}.
$$

Find the capacity $C$ for

(a) $\rho = 1$.

(b) $\rho = 0$.

(c) $\rho = -1$.

**Problem 2.** Consider a pair of parallel Gaussian channels, i.e.,

$$
\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix},
$$

where

$$
\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} \sim \mathcal{N} \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} \right),
$$

and there is a power constraint $E(X_1^2 + X_2^2) \leq 2P$. Assume that $\sigma_1^2 > \sigma_2^2$.

(a) Suppose we use the capacity achieving distribution as input. At what power does the channel stop behaving like a single channel with noise variance $\sigma_2^2$, and begin behaving like a pair of channels?

(b) Let $C_1(P)$ be the capacity of the pair of gaussian channels when the input is constrained to have a power not exceeding $2P$. Let $C_2(P) = I(X_1, X_2; Y_1, Y_2)$ when both $X_1$ and $X_2$ are independent gaussian random variables with variance equal to $P$. Show that $C_1(P) - C_2(P)$ tends to zero as $P/\sigma_1^2$ tends to infinity.
Problem 3. Consider a vector Gaussian channel described as follows:

\[ Y_1 = x + Z_1 \]
\[ Y_2 = Z_2 \]

where \( x \) is the input to the channel constrained in power to \( P \); \( Z_1 \) and \( Z_2 \) are jointly Gaussian random variables with \( E[Z_1] = E[Z_2] = 0, E[Z_1^2] = E[Z_2^2] = \sigma^2 \) and \( E[Z_1Z_2] = \rho \sigma^2 \), with \( \rho \in [-1, 1] \), and independent of the channel input.

(a) Consider a receiver that discards \( Y_2 \) and decodes the message based only on \( Y_1 \). What rates are achievable with such a receiver?

(b) Consider a receiver that forms \( Y = Y_1 - \rho Y_2 \), and decodes the message based only on \( Y \). What rates are achievable with such a receiver?

(c) Find the capacity of the channel and compare it to the part (b).

Problem 4.

a) Let \( x^* \) be the most probable letter of a finite source \( X \), i.e. \( P(x^*) \geq P(x) \), for all \( x \in X \). Show that
\[ H(X) \geq \log\left(\frac{1}{P(x^*)}\right). \]

b) [Fano’s Inequality] Assume that \( X \) generates a letter and we want to estimate the outcome of \( X \) by observing random variable \( Y \) which is related to \( X \) by the conditional distribution \( p(y|x) \). From \( Y \), we calculate a function \( g(Y) = \hat{X} \), where \( \hat{X} \) is an estimate of \( X \). Let \( P_e \) be the error probability of estimation defined as \( P_e = P\{\hat{X} \neq X\} \). Prove that
\[ H(X \mid Y) \leq H(P_e) + P_e \log(|X| - 1), \]
where \(|X|\) denotes the number of letters in the alphabet \( X \).

c) [Fano’s Inverse Inequality] Assume that we use a Maximum A Posteriori estimator, i.e. for an observation \( y \),
\[ \hat{x} = g(y) = \arg \max_{x \in X} p(x|y). \]

Prove that
\[ P_e \leq 1 - 2^{-H(X \mid Y)}. \]

Hint: use part (a) and note that \( \sum_i p_i 2^{-u_i} \geq 2^{-\sum_i p_i u_i} \).

Problem 5. If \( X_1, X_2, \ldots \) are random variables such that \( E[X_i \mid X_{i-1}] = X_{i-1} \), show that \( E[(X_{i+1} - X_i)(X_{j+1} - X_j)] = 0 \) if \( i \neq j \).

Problem 6. Compute the transition probabilities of the synthesized + and − channels after applying the basic polarization transformations you have seen in class starting from a binary symmetric channel with crossover probability \( \epsilon \).