Solutions 4

1. a) Let us compute

$$\sum_{j,k=0}^{n} c_j\, c_k\, m_{j+k} = \sum_{j,k=0}^{n} c_j\, c_k \int_{\mathbb{R}} x^{j+k}\, d\mu(x) = \int_{\mathbb{R}} \left(\sum_{j=0}^{n} c_j x^j \right)^2 d\mu(x) \geq 0$$

b) i), ii) and iv) are sequences of moments: i) corresponds to the uniform distribution on $[0,1]$.

iii) corresponds to $\mu = \delta_e$, that is, the "distribution" concentrated on the single point e.

iv) corresponds to the log-normal distribution with pdf $p_\mu(x) = \frac{1}{\sqrt{2\pi}x} \exp(-(\log x)^2/2)$, $x > 0$. But as the sequence $(m_k,\, k \geq 0)$ does not satisfy Carleman's condition, there are other distributions with the same sequence of moments, such as e.g. the discrete version of the log-normal distribution defined as $\mu(\{e^j\}) = C \exp(-j^2/2)$, $j \in \mathbb{Z}$, with $C^{-1} = \sum_{j \in \mathbb{Z}} \exp(-j^2/2)$.

ii) does not corresspond to a distribution, as the correspoding matrices $A^{(n)}$ are not positive semi-definite for all values of n. For example, $A^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 4 \end{pmatrix}$ has eigenvalues $\lambda_\pm = 2 \pm \sqrt{5}$, therefore is not positive semi-definite.

2. a) By the indicated change of variable, we obtain:

$$\begin{aligned} m_k &= \int_0^4 x^{2k}\, \frac{1}{\pi} \sqrt{\tfrac{1}{x} - \tfrac{1}{4}}\, dx = \frac{1}{\pi} \int_0^{\pi/2} 4^k\, \sin(t)^{2k} \sqrt{\tfrac{1}{4\sin(t)^2} - \tfrac{1}{4}}\, 8 \sin(t)\cos(t)\, dt \\ &= \frac{4^{k+1}}{\pi} \int_0^{\pi/2} \sin(t)^{2k}\, \cos(t)^2\, dt = \frac{4^{k+1}}{\pi} \left(\int_0^{\pi/2} \sin(t)^{2k}\, dt - \int_0^{\pi/2} \sin(t)^{2(k+1)}\, dt \right). \end{aligned}$$

By integration by parts (with $u(t) = \sin^{2k-1}(t)$ and $v(t) = \sin(t)$), we have

$$a_{k+1} := \int_0^{\pi/2} \sin(t)^{2(k+1)}\, dt = (2k+1) \int_0^{\pi/2} \sin(t)^{2k}\, \cos^2(t)\, dt = (2k+1)\,(a_k - a_{k+1}),$$

so

$$a_{k+1} = \frac{2k+1}{2k+2}\, a_k = \ldots = \frac{(2k+1)\cdots 3 \cdot 1}{(2k+2)\cdots 4 \cdot 2}\, a_0 = \frac{(2k+1)!/(2^k\, k!)}{2^{k+1}\,(k+1)!}\, a_0 = \frac{(2k+1)!}{2^{2k+1}\, k!(k+1)!}\, a_0.$$

Since $a_0 = \frac{\pi}{2}$, we finally obtain

$$m_k = \frac{4^{k+1}}{\pi}\, \frac{a_{k+1}}{2k+1} = \frac{(2k)!}{k!(k+1)!} = \frac{1}{k+1} \begin{pmatrix} 2k \\ k \end{pmatrix}.$$

Carleman's condition is satisfied, since

$$m_k \leq \frac{(2k)!}{(k!)^2} \leq \frac{((2k)(2k-2)\cdots 2)^2}{(k!)^2} = \frac{(2^k\, k!)^2}{(k!)^2} \leq 4^k.$$

An easier way to see this is to notice that μ has compact support $[0,4]$.

b) By the change of variable $y = x^\lambda$, we have

$$\int_{\mathbb{R}} x^k \, d\mu(x) = c_\lambda \int_0^\infty x^k \exp(-x^\lambda) \, dx = c_\lambda \, \lambda \int_0^\infty y^{k/\lambda} \, e^{-y} \, y^{1/\lambda - 1} \, dy = c_\lambda \, \lambda \, \Gamma((k+1)/\lambda),$$

where Γ is the Euler Gamma function. Using the approximation $\Gamma(x+1) \sim [x]!$, we see that

$$m_k = \int_{\mathbb{R}} x^k \, d\mu(x) \sim [\tfrac{k}{\lambda}]!$$

so by Stirling's formula ($\log(k!) \sim k \log k$),

$$\limsup_{k\to\infty} \frac{1}{2k} \, (m_{2k})^{\frac{1}{2k}} \sim \limsup_{k\to\infty} \frac{1}{2k} \, e^{\frac{1}{\lambda}\log(2k/\lambda)} \sim \limsup_{k\to\infty} \frac{1}{2k} \, (2k/\lambda)^{\frac{1}{\lambda}} < \infty$$

if and only if $\lambda \geq 1$. We can deduce the following rule of thumb from the preceding argument: a distribution is uniquely determined by its moments as long as its tail is not heavier than the exponential e^{-x}.

3. a) We have:

$$\text{Re } g_\mu(u+iv) = \int_{\mathbb{R}} \frac{x-u}{(x-u)^2+v^2} \, d\mu(x) \quad \text{and} \quad \text{Im } g_\mu(u+iv) = \int_{\mathbb{R}} \frac{v}{(x-u)^2+v^2} \, d\mu(x).$$

b*) The analyticity of g_μ on $\mathbb{C}\backslash\mathbb{R}$ follows from the analyticity of $z \mapsto \frac{1}{x-z}$ on $\mathbb{C}\backslash\mathbb{R}$ and the use of the dominated convergence theorem.

c) If $v > 0$, then Im $g_\mu(u+iv)$ is clearly positive by the above formula.

d) We have:

$$v^2 \, |g_\mu(iv)|^2 = \left(\int_{\mathbb{R}} \frac{v(x-u)}{(x-u)^2+v^2} \, d\mu(x) \right)^2 + \left(\int_{\mathbb{R}} \frac{v^2}{(x-u)^2+v^2} \, d\mu(x) \right)^2.$$

By the dominated convergence theorem, the first term on the right-hand side converges to 0 as $v \to +\infty$ and the second term converges to 1.

e) This is a straightforward computation.

4. a) We have

$$\begin{aligned} \frac{1}{\pi} \lim_{\varepsilon\downarrow 0} \int_a^b \text{Im } g_\mu(x+i\varepsilon) \, dx &= \frac{1}{\pi} \lim_{\varepsilon\downarrow 0} \int_a^b \left(\int_{\mathbb{R}} \frac{\varepsilon}{(y-x)^2+\varepsilon^2} \, d\mu(y) \right) dx \\ &= \frac{1}{\pi} \lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}} \left(\int_a^b \frac{\varepsilon}{(y-x)^2+\varepsilon^2} \, dx \right) d\mu(y) \\ &= \frac{1}{\pi} \lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}} \text{arctg} \left(\frac{y-x}{\varepsilon} \right) \Bigg|_{x=a}^{x=b} d\mu(y). \end{aligned}$$

Since

$$\lim_{\varepsilon\downarrow 0} \text{arctg} \left(\frac{y-x}{\varepsilon} \right) \Bigg|_{x=a}^{x=b} = \begin{cases} \pi, & \text{if } a < y < b, \\ \frac{\pi}{2}, & \text{if } y = a \text{ or } b \\ 0, & \text{otherwise} \end{cases} = \pi \, (1_{]a,b[}(y) + \frac{1}{2} \, 1_{\{a,b\}}(y)),$$

we conclude by the dominated convergence theorem that for any $a < b$ continuity points of F_μ,

$$\frac{1}{\pi} \lim_{\varepsilon\downarrow 0} \int_a^b \text{Im } g_\mu(x+i\varepsilon) \, dx = F_\mu(b) - F_\mu(a).$$

b) Assuming that μ admits a pdf p_μ, the same computation as above leads to

$$\begin{aligned}
\frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\int_a^b \mathrm{Im}\; g_\mu(x+i\varepsilon)\,dx &= \frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\int_a^b \left(\int_{\mathbb{R}} \frac{\varepsilon}{(y-x)^2+\varepsilon^2}\, p_\mu(y)\,dy\right)\,dx \\
&= \int_{\mathbb{R}} (1_{]a,b[}(y) + \frac{1}{2}\,1_{\{a,b\}}(y))\, p_\mu(y)\,dy = \int_a^b p_\mu(y)\,dy.
\end{aligned}$$

c) The two solutions to the quadratic equation are

$$g_\pm(z) = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{1}{z}}$$

and for $\mathrm{Im}\; z > 0$, only g_+ satsifies $\mathrm{Im}\; g_+(z) > 0$. Therefore,

$$\begin{aligned}
p_\mu(x) &= \frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\mathrm{Im}\;(g_+(z)) = \frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\mathrm{Im}\left(\sqrt{\frac{1}{4} - \frac{1}{x+i\varepsilon}}\right) \\
&= \frac{1}{\pi}\,\mathrm{Im}\left(\sqrt{\frac{1}{4} - \frac{1}{x}}\right) = \frac{1}{\pi}\sqrt{\frac{1}{x} - \frac{1}{4}}\,1_{\{0<x\leq 4\}}.
\end{aligned}$$