Solutions 4

1. a) Let us compute
\[ \sum_{j,k=0}^{n} c_j c_k m_{j+k} = \sum_{j,k=0}^{n} c_j c_k \int_{\mathbb{R}} x^{j+k} \, d\mu(x) = \int_{\mathbb{R}} \left( \sum_{j=0}^{n} c_j x^j \right)^2 \, d\mu(x) \geq 0 \]

b) i), ii) and iv) are sequences of moments: i) corresponds to the uniform distribution on \([0, 1]\).

iii) corresponds to \(\mu = \delta_e\), that is, the “distribution” concentrated on the single point \(e\).

iv) corresponds to the log-normal distribution with pdf \(p_\mu(x) = \frac{1}{\sqrt{2\pi x}} \exp(-\log(x)^2/2)\), \(x > 0.\) But as the sequence \((m_k, k \geq 0)\) does not satisfy Carleman’s condition, there are other distributions with the same sequence of moments, such as e.g. the discrete version of the log-normal distribution defined as \(\mu(\{e^j\}) = C \exp(-j^2/2), j \in \mathbb{Z}\), with \(C^{-1} = \sum_{j \in \mathbb{Z}} \exp(-j^2/2)\).

ii) does not correspond to a distribution, as the corresponding matrices \(A^{(n)}\) are not positive semi-definite for all values of \(n\). For example, \(A^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 4 \end{pmatrix}\) has eigenvalues \(\lambda_{\pm} = 2 \pm \sqrt{5}\), therefore is not positive semi-definite.

2. a) By the indicated change of variable, we obtain:
\[ m_k = \int_0^4 x^{2k} \frac{1}{\pi} \sqrt{\frac{1}{x} - 1} \, dx = \frac{1}{\pi} \int_0^{\pi/2} 4^k \sin(t)^{2k} \sqrt{\frac{1}{4 \sin(t)^2} - \frac{1}{4}} \, 8 \sin(t) \cos(t) \, dt \]
\[ = \frac{4^{k+1}}{\pi} \int_0^{\pi/2} \sin(t)^{2k} \cos(t)^2 \, dt = \frac{4^{k+1}}{\pi} \left( \int_0^{\pi/2} \sin(t)^{2k} \, dt - \int_0^{\pi/2} \sin(t)^{2(k+1)} \, dt \right). \]

By integration by parts (with \(u(t) = \sin^{2k-1}(t)\) and \(v(t) = \sin(t)\)), we have
\[ a_{k+1} := \int_0^{\pi/2} \sin(t)^{2(k+1)} \, dt = (2k + 1) \int_0^{\pi/2} \sin(t)^{2k} \cos^2(t) \, dt = (2k + 1) (a_k - a_{k+1}), \]
so
\[ a_{k+1} = \frac{2k + 1}{2k + 2} a_k = \ldots = \frac{(2k + 1) \cdots 3 \cdot 1}{(2k + 2) \cdots 4 \cdot 2} a_0 = \frac{(2k + 1)!/(2^k k!)}{2k+1 (k+1)!} a_0 = \frac{(2k + 1)!}{2^{2k+1} k! (k+1)!} a_0. \]

Since \(a_0 = \pi^2\), we finally obtain
\[ m_k = \frac{4^{k+1}}{\pi} \frac{a_{k+1}}{2k + 1} = \frac{(2k)!}{k! (k+1)!} = \frac{1}{k+1} \binom{2k}{k}. \]

Carleman’s condition is satisfied, since
\[ m_k \leq \frac{(2k)!}{(k!)^2} \leq \frac{((2k)(2k-2) \cdots 2)^2}{(k!)^2} = \frac{(2k)!^2}{(k!)^2} \leq 4^k. \]

An easier way to see this is to notice that \(\mu\) has compact support \([0, 4]\).
b) By the change of variable $y = x^\lambda$, we have
\[ \int_{\mathbb{R}} x^k d\mu(x) = c_\lambda \int_0^\infty x^k \exp(-x^\lambda) dx = c_\lambda \lambda \int_0^\infty y^{k/\lambda} e^{-y} y^{1/\lambda - 1} dy = c_\lambda \lambda \Gamma((k + 1)/\lambda), \]
where $\Gamma$ is the Euler Gamma function. Using the approximation $\Gamma(x + 1) \sim [x]!$, we see that
\[ m_k = \int_{\mathbb{R}} x^k d\mu(x) \sim [k]! \]
so by Stirling’s formula ($\log(k!) \sim k \log k$),
\[ \limsup_{k \to \infty} \frac{1}{2k} (m_{2k})^{1/2k} \sim \limsup_{k \to \infty} \frac{1}{2k} e^{\frac{1}{2} \log(2k/\lambda)} \sim \limsup_{k \to \infty} \frac{1}{2k} (2k/\lambda)^{1/2} < \infty \]
if and only if $\lambda \geq 1$. We can deduce the following rule of thumb from the preceding argument: a distribution is uniquely determined by its moments as long as its tail is not heavier than the exponential $e^{-x}$.

3. a) We have:
\[ \text{Re } g_\mu(u + iv) = \int_{\mathbb{R}} \frac{x - u}{(x - u)^2 + v^2} d\mu(x) \quad \text{and} \quad \text{Im } g_\mu(u + iv) = \int_{\mathbb{R}} \frac{v}{(x - u)^2 + v^2} d\mu(x). \]

b* The analyticity of $g_\mu$ on $\mathbb{C}\\setminus\mathbb{R}$ follows from the analyticity of $z \mapsto \frac{1}{z^2}$ on $\mathbb{C}\\setminus\mathbb{R}$ and the use of the dominated convergence theorem.

c) If $v > 0$, then $\text{Im } g_\mu(u + iv)$ is clearly positive by the above formula.

d) We have:
\[ v^2 |g_\mu(iv)|^2 = \left( \int_{\mathbb{R}} \frac{v(x - u)}{(x - u)^2 + v^2} d\mu(x) \right)^2 + \left( \int_{\mathbb{R}} \frac{v^2}{(x - u)^2 + v^2} d\mu(x) \right)^2. \]

By the dominated convergence theorem, the first term on the right-hand side converges to 0 as $v \to +\infty$ and the second term converges to 1.

e) This is a straightforward computation.

4. a) We have
\[ \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \int_a^b \text{Im } g_\mu(x + i\varepsilon) \, dx = \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \int_a^b \left( \int_{\mathbb{R}} \frac{\varepsilon}{(y - x)^2 + \varepsilon^2} d\mu(y) \right) \, dx \]
\[ = \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}} \left( \int_a^b \frac{\varepsilon}{(y - x)^2 + \varepsilon^2} \, dx \right) d\mu(y) \]
\[ = \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}} \arctg \left( \frac{y - x}{\varepsilon} \right) \bigg|_{x=a}^{x=b} d\mu(y). \]

Since
\[ \lim_{\varepsilon \downarrow 0} \arctg \left( \frac{y - x}{\varepsilon} \right) \bigg|_{x=a}^{x=b} = \begin{cases} \pi, & \text{if } a < y < b, \\ \frac{\pi}{2}, & \text{if } y = a \text{ or } b = \pi \left(1_{[a,b]}(y) + \frac{1}{2} 1_{(a,b)}(y)\right), \\ 0, & \text{otherwise} \end{cases} \]
we conclude by the dominated convergence theorem that for any $a < b$ continuity points of $F_\mu$,
\[ \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \int_a^b \text{Im } g_\mu(x + i\varepsilon) \, dx = F_\mu(b) - F_\mu(a). \]
b) Assuming that $\mu$ admits a pdf $p_\mu$, the same computation as above leads to

$$
\frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \int_a^b \text{Im} \, g_\mu(x + i\varepsilon) \, dx = \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \int_a^b \left( \int_R \frac{\varepsilon}{(y-x)^2 + \varepsilon^2} p_\mu(y) \, dy \right) \, dx \\
= \int_R (1_{[a,b]}(y) + \frac{1}{2} 1_{(a,b)}(y)) p_\mu(y) \, dy = \int_a^b p_\mu(y) \, dy.
$$

c) The two solutions to the quadratic equation are

$$g_\pm(z) = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{1}{z}}$$

and for $\text{Im} \, z > 0$, only $g_+$ satisfies $\text{Im} \, g_+(z) > 0$. Therefore,

$$p_\mu(x) = \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \text{Im} \, (g_+(z)) = \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \left( \sqrt{\frac{1}{4} - \frac{1}{x + i\varepsilon}} \right) = \frac{1}{\pi} \text{Im} \, \left( \sqrt{\frac{1}{4} - \frac{1}{x}} \right) = \frac{1}{\pi} \sqrt{\frac{1}{x} - \frac{1}{4}} 1_{\{0 < x \leq 4\}}.$$. 