Solutions 2

1. a) The joint distribution of the entries of H is given by

$$p_H(H) = \prod_{j,k=1}^{2} \frac{1}{\sqrt{2\pi}} \exp(-h_{jk}^2/2) = \frac{1}{4\pi^2} \exp(-Tr(HH^T)/2)$$

b) Componentwise, the change of variable $H = LQ$ reads:

$$h_{11} = a\cos(u), \quad h_{12} = a\sin(u), \quad h_{21} = b\cos(u) - c\sin(u), \quad h_{22} = b\sin(u) + c\cos(u)$$

so the Jacobian is given by

$$J = \det \begin{pmatrix} \cos(u) & \sin(u) & 0 & 0 \\ 0 & 0 & \cos(u) & \sin(u) \\ 0 & 0 & -\sin(u) & \cos(u) \\ -a\sin(u) & a\cos(u) & -b\sin(u) - c\cos(u) & b\cos(u) - c\sin(u) \end{pmatrix} = a$$

Therefore,

$$p_{L,Q}(a,b,c,u) = \frac{a}{4\pi^2} \exp(-(a^2 + 2b^2 + c^2)/2)$$

The 4 random variables a, b, c, u are therefore independent and u is uniformly distributed on $[0, 2\pi]$, so

$$p_L(a,b,c) = \frac{a}{2\pi} \exp(-(a^2 + 2b^2 + c^2)/2)$$

NB: There is a little bug in the above argument, as any matrix H is not necessarily of the form LQ with $l_{jj} \geq 0$ for all j and Q orthogonal with $\det(Q) = +1$. In order to be completely rigorous, we should first consider the change of variable $H = LQ$ with no restriction on the diagonal entries of L (other than l_{jj} being real) and Q being such that $\det(Q) = +1$. A computation totally similar to the above computation leads to the joint distribution

$$P_{L,Q}(L,Q) = C \exp(-\text{Tr}(LL^T)/2)$$

Choose now a matrix $\Sigma = \text{diag}(\pm 1, \ldots, \pm 1)$ such that $(L\Sigma)_{jj} \geq 0$ for all j, we obtain

$$H = LQ = (L\Sigma)(\Sigma Q)$$

One can check that Σ is uniformly distributed, so ΣQ is also uniformly distributed on the set of orthogonal matrices (and therefore independent of $L\Sigma$).

c) Componentwise, the change of variable $W = LL^T$ reads

$$w_{11} = a^2, \quad w_{12} = ab, \quad w_{22} = b^2 + c^2$$

so the (reverse) Jacobian is given by

$$J = \det \begin{pmatrix} 2a & b & 0 \\ 0 & a & 2b \\ 0 & 0 & 2c \end{pmatrix} = 4a^2 c$$

and

$$\begin{aligned} p_W(W) &= \frac{a}{2\pi} \exp(-(a^2 + 2b^2 + c^2)/2) \frac{1}{4a^2 c} = \frac{1}{8\pi ac} \exp(-(a^2 + 2b^2 + c^2)/2) \\ &= \frac{1}{8\pi \det(L)} \exp(-\text{Tr}(LL^T)/2) = \frac{1}{8\pi\sqrt{\det W}} \exp(-\text{Tr}(W)/2) \end{aligned}$$

2. a) As Q is positive (semi-)definite, it holds that $x^*Qx \geq 0$ for all $x \in \mathbb{C}^n$, so in particular also for $x = H^*y$, where $y \in \mathbb{C}^n$ is arbitrary. So $y^*HQH^*y \geq 0$ for all $y \in \mathbb{C}^n$, which is saying that HQH^* is positive semi-definite.

b) Let $Q = VMV^*$ be the eigenvalue decomposition of Q. We know that H and HV have the same distribution, for any unitary matrix V. So $W = (HV)M(HV)^*$ and HMH^* also have the same distribution.

c) $\widetilde{h}_{jk} = h_{jk}\sqrt{\mu_k}$, so

$$p_{\widetilde{h}_{jk}}(z) = \frac{1}{\mu_k}\, p_{h_{jk}}(z/\sqrt{\mu_k}) = \frac{1}{\pi\,\mu_k}\, \exp\left(-\frac{|z|^2}{\mu_k}\right)$$

and

$$p_{\widetilde{H}}(\widetilde{H}) = \prod_{j,k=1}^{n} \frac{1}{\pi\mu_k}\, \exp\left(-\frac{|\widetilde{h}_{jk}|^2}{\mu_k}\right) = \frac{C_n}{(\det M)^n}\, \exp(-\mathrm{Tr}(\widetilde{H}M^{-1}\widetilde{H}^*))$$

where $C_n = 1/\pi^{n^2}$ is a constant.

d) In the course, we have seen that the Jacobian of the transfromation $\widetilde{H} \mapsto \widetilde{W}$ is a constant (remember that we are in the case where $\widetilde{H}$ is square here), so

$$p_{\widetilde{W}}(\widetilde{W}) = \frac{C_n}{(\det M)^n}\, \exp(-\mathrm{Tr}(M^{-1}\widetilde{W}))$$

e) Let $\widetilde{W} = U\Lambda U^*$ be the eigenvalue decomposition of $\widetilde{W}$. By the course, we have

$$p_{\Lambda,U}(\Lambda, U) = p_{\widetilde{W}}(U\Lambda U^*)\, |J(\Lambda, U)|$$

where $J(\Lambda, U) = \Delta(\Lambda)^2 = \prod_{j<k}(\lambda_k - \lambda_j)^2$. Therefore,

$$p_{\Lambda,U}(\Lambda, U) = C_n\, \frac{\Delta(\Lambda)^2}{(\det M)^n}\, \exp(-\mathrm{Tr}(M^{-1}U\Lambda U^*))$$

and consequently,

$$p_\Lambda(\Lambda) = C_n\, \frac{\Delta(\Lambda)^2}{(\det M)^n} \int_{\mathcal{U}_n} dU\, \exp(-\mathrm{Tr}(M^{-1}U\Lambda U^*))$$

where $\mathcal{U}_n$ is the group of $n \times n$ unitary matrices and dU is the Haar measure on this group. This integral can be further computed via the *Harish-Chandra formula*:

$$\int_{\mathcal{U}_n} dU\, \exp(-\mathrm{Tr}(M^{-1}U\Lambda U^*)) = C_n\, \frac{\det\left(\{\exp(-\lambda_j/\mu_k)\}_{j,k=1}^n\right)}{\Delta(\Lambda)\,\Delta(-M^{-1})}$$

Noticing that

$$\Delta(-M^{-1}) = \prod_{j<k}\left(\frac{1}{\mu_j} - \frac{1}{\mu_k}\right) = \prod_{j<k}\left(\frac{\mu_k - \mu_j}{\mu_j\,\mu_k}\right) = \frac{\Delta(M)}{(\det M)^{n-1}}$$

we finally obtain

$$p_\Lambda(\Lambda) = \frac{C_n}{\det M}\, \frac{\Delta(\Lambda)}{\Delta(M)}\, \det\left(\{\exp(-\lambda_j/\mu_k)\}_{j,k=1}^n\right)$$