Solutions 1

1. First notice that there is nothing to prove if det($A$) = 0, so we may assume that det($A$) > 0.
Let $X$ be a complex Gaussian vector with covariance matrix $A$. Then by the hint,
\[
\log \det(\pi eA) \leq \sum_{j=1}^{n} \log(\pi e a_{jj})
\]
which implies Hadamard’s inequality (simplifying by $\pi e$).

2. We need to show that for any pair of positive definite matrices $A$, $B$ and any $p \in [0, 1],$
\[
\log \det(pA + (1 - p)B) \geq p \log \det(A) + (1 - p) \log \det(B)
\]
Following the hint, let $X$, $Y$ be independent Gaussian vectors with covariance matrices $A$, $B$, and let $\Theta$ be a random variable independent of both $X$ and $Y$ such that $P(\Theta = 1) = p = 1 - P(\Theta = 0)$. Let also $Z$ be the random vector such that $Z = X$ if $\Theta = 1$ and $Z = Y$ if $\Theta = 0$. The covariance matrix of $Z$ is $pA + (1 - p)B$, so
\[
h(Z) \leq \log \det(\pi e (pA + (1 - p)B))
\]
(NB: this inequality is an equality when $Z$ is Gaussian, but here, $Z$ is not Gaussian). On the other hand,
\[
h(Z) \geq h(Z|\Theta) = p h(X) + (1 - p) h(Y) = p \log \det(\pi eA) + (1 - p) \log \det(\pi eB)
\]
which gives the desired inequality (simplifying by $\pi e$ again).

3. a) If the channel coefficients are i.i.d., then $H$ and $H\Pi$ share the same distribution for any permutation matrix, so
\[
\psi(\Pi\Pi^*) = E_H(\log \det(I + (H\Pi)Q(H\Pi)^*)) = \psi(Q)
\]
b) By Ex. 2, $Q \mapsto \psi(Q)$ is concave, so letting $P_n$ be the set of all permutation matrices and using part a), we obtain, for any $Q > 0,$
\[
\psi(Q) = \frac{1}{n!} \sum_{\Pi \in P_n} \psi(\Pi\Pi^*) \leq \psi(\overline{Q})
\]
where
\[
\overline{Q} = \frac{1}{n!} \sum_{\Pi \in P_n} \Pi\Pi^*
\]
Notice that $\overline{Q} \geq 0$ and $\text{Tr}(\overline{Q}) = \text{Tr}(Q) \leq P$. Also, observe that all diagonal coefficients of $\overline{Q}$ are equal, and the same holds for all non-diagonal coefficients as well. Therefore, $\overline{Q}$ satisfies the constraints and has the form given in the problem set (choosing $\text{Tr}(\overline{Q}) = P$). As the above inequality holds for any $Q$ satisfying the constraints, we obtain that $Q_{opt}$ is of the form given in the problem set. The fact that $-\frac{1}{n-1} \leq c \leq 1$ is due to the constraint that $Q_{opt} \geq 0.$
4. a) If the channel coefficients are independent and such that for all \( j, k \), \(-h_{j,k}\) has the same distribution as \( h_{j,k} \), then \( H \) and \( H\Sigma \) share the same distribution for any matrix \( \Sigma = \text{diag}(\pm 1, \ldots, \pm 1) \), so, as above,

\[
\psi(\Sigma Q \Sigma^*) = E_H(\log \det(I + (H\Sigma)Q(H\Sigma)^*)) = \psi(Q)
\]

b) By Ex. 2, \( Q \mapsto \psi(Q) \) is concave, so letting \( S_n \) be the set of all matrices \( \Sigma \) and using part a), we obtain, for any \( Q > 0 \),

\[
\psi(Q) = \frac{1}{2^n} \sum_{\Sigma \in S_n} \psi(\Sigma Q \Sigma^*) \leq \psi(\hat{Q})
\]

where

\[
\hat{Q} = \frac{1}{2^n} \sum_{\Sigma \in S_n} \Sigma Q \Sigma^*
\]

Notice that \( \hat{Q} \geq 0 \) and \( \text{Tr}(\hat{Q}) = \text{Tr}(Q) \leq P \). Also, observe that all non-diagonal coefficients of \( \hat{Q} \) are zero. Therefore, \( \hat{Q} \) satisfies the constraints and is diagonal. As the above inequality holds for any \( Q \) satisfying the constraints, we obtain that \( Q_{\text{opt}} \) is diagonal.