Solutions 1

1. First notice that there is nothing to prove if $\det(A) = 0$, so we may assume that $\det(A) > 0$. Let X be a complex Gaussian vector with covariance matrix A. Then by the hint,

$$\log\det(\pi e A) \leq \sum_{j=1}^{n} \log(\pi e a_{jj})$$

which implies Hadamard's inequality (simplifying by πe).

2. We need to show that for any pair of positive definite matrices A, B and any $p \in [0,1]$,

$$\log\det(p\,A + (1-p)\,B) \geq p\log\det(A) + (1-p)\,\log\det(B)$$

Following the hint, let X, Y be independent Gaussian vectors with covariance matrices A, B, and let Θ be a random variable independent of both X and Y such that $\mathbb{P}(\Theta = 1) = p = 1 - \mathbb{P}(\Theta = 0)$. Let also Z be the random vector such that $Z = X$ if $\Theta = 1$ and $Z = Y$ if $\Theta = 0$. The covariance matrix of Z is $p\,A + (1-p)\,B$, so

$$h(Z) \leq \log\det(\pi e\,(p\,A + (1-p)\,B))$$

(NB: this inequality is an equality when Z is Gaussian, but here, Z is *not* Gaussian). On the other hand,

$$h(Z) \geq h(Z|\Theta) = p\,h(X) + (1-p)\,h(Y) = p\,\log\det(\pi e A) + (1-p)\,\log\det(\pi e B)$$

which gives the desired inequality (simplifying by πe again).

3. a) If the channel coefficients are i.i.d., then H and $H\Pi$ share the same distribution for any permutation matrix, so

$$\psi(\Pi Q \Pi^*) = \mathbb{E}_H(\log\det(I + (H\Pi)Q(H\Pi)^*)) = \psi(Q)$$

b) By Ex. 2, $Q \mapsto \psi(Q)$ is concave, so letting $\mathcal{P}_n$ be the set of all permutation matrices and using part a), we obtain, for any $Q > 0$,

$$\psi(Q) = \frac{1}{n!}\sum_{\Pi \in \mathcal{P}_n} \psi(\Pi Q \Pi^*) \leq \psi(\overline{Q})$$

where

$$\overline{Q} = \frac{1}{n!}\sum_{\Pi \in \mathcal{P}_n} \Pi Q \Pi^*$$

Notice that $\overline{Q} \geq 0$ and $\mathrm{Tr}(\overline{Q}) = \mathrm{Tr}(Q) \leq P$. Also, observe that all diagonal coefficients of $\overline{Q}$ are equal, and the same holds for all non-diagonal coefficients as well. Therefore, $\overline{Q}$ satisfies the constraints and has the form given in the problem set (choosing $\mathrm{Tr}(\overline{Q}) = P$). As the above inequality holds for *any* Q satisfying the constraints, we obtain that Q_{opt} is of the form given in the problem set. The fact that $-\frac{1}{n-1} \leq c \leq 1$ is due to the constraint that $Q_{opt} \geq 0$.

4. a) If the channel coefficients are independent and such that for all j, k, $-h_{j,k}$ has the same distribution as h_{jk}, then H and $H\Sigma$ share the same distribution for any matrix $\Sigma = \mathrm{diag}(\pm 1, \ldots, \pm 1)$, so, as above,

$$\psi(\Sigma Q \Sigma^*) = \mathbb{E}_H(\log\det(I + (H\Sigma)Q(H\Sigma)^*)) = \psi(Q)$$

b) By Ex. 2, $Q \mapsto \psi(Q)$ is concave, so letting $\mathcal{S}_n$ be the set of all matrices Σ and using part a), we obtain, for any $Q > 0$,

$$\psi(Q) = \frac{1}{2^n} \sum_{\Sigma \in \mathcal{S}_n} \psi(\Sigma Q \Sigma^*) \leq \psi(\widehat{Q})$$

where

$$\widehat{Q} = \frac{1}{2^n} \sum_{\Sigma \in \mathcal{S}_n} \Sigma Q \Sigma^*$$

Notice that $\hat{Q} \geq 0$ and $\mathrm{Tr}(\widehat{Q}) = \mathrm{Tr}(Q) \leq P$. Also, observe that all non-diagonal coefficients of $\widehat{Q}$ are zero. Therefore, $\widehat{Q}$ satisfies the constraints and is diagonal. As the above inequality holds for *any* Q satisfying the constraints, we obtain that Q_{opt} is diagonal.