Random matrices and communication systems: WEEK 5

1 Wishart random matrices: joint distribution of the entries

1.1 Complex case

Let H be an $n \times m$ random matrix with i.i.d.$\sim \mathcal{N}_{\mathbb{C}}(0,1)$ entries and let W be the $n \times n$ matrix defined as $W = HH^*$. W is a positive semi-definite matrix, as $x^* W x = \|H^* x\|^2 \geq 0$ for all $x \in \mathbb{C}^n$. So by the spectral theorem, there exist U $n \times n$ unitary (i.e. $UU^* = I$) and $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$ such that $\lambda_j \geq 0$ for all j and $W = U \Lambda U^*$. Ultimately, we are interested in computing the joint distribution $p(\lambda_1, \ldots, \lambda_n)$ of the eigenvalues of W (for fixed values of both n and m), as well as its marginals. This however requires a first step, namely to compute the joint distribution of the entries of W, which is the purpose of the present lecture.

Remark. If $m < n$, then $\text{rank}(W) \leq \text{rank}(H) \leq \min(m,n) = m < n$, which is saying the the matrix W is rank-deficient and admits at least $n-m$ zero eigenvalues. So in this case, the joint distribution of $\lambda_1, \ldots, \lambda_n$ is singular. Instead, we may as well consider the $m \times m$ matrix $\widetilde{W} = H^* H$, which has the same non-zero eigenvalues as W, and look for the joint distribution of its eigenvalues $\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_m$. In the following, we therefore restrict ourselves without loss of generality to the case where $m \geq n$.

Let us now describe the various steps that lead to the joint distribution of the entries of W.

Joint distribution of the entries of H. This is an easy computation. By independence, we have

$$p_H(H) = \prod_{j,k=1}^{n,m} \frac{1}{\pi} \exp\left(-|h_{jk}|^2\right) = \frac{1}{\pi^{nm}} \exp\left(-\sum_{j,k=1}^{n,m} |h_{jk}|^2\right) = \frac{1}{\pi^{nm}} \exp\left(-\text{Tr}\,(HH^*)\right)$$

LQ decomposition of H and Choleski decomposition of W. Let us recall the following fact:

Any $n \times m$ complex-valued matrix H may be decomposed into $H = LQ$, where L is an $n \times n$ lower-triangular matrix such that $l_{jj} \geq 0$ for all j and Q is an $n \times m$ matrix such that $QQ^* = I_n$ (i.e. Q is a submatrix of an $m \times m$ unitary matrix).

Consequently, $W = HH^* = LQQ^*L^* = LL^*$, which is the Cholesky decomposition of W.

The strategy now is, starting from the expression for $p_H(H)$, to compute $p_L(L)$ and then $p_W(W)$.

Joint distribution of the entries of L. The relation $H = LQ$ can be seen as a change of variables $H \mapsto (L, Q)$. Let us first check how many free real parameters there are on each side of the equality. On the left-hand side, H has clearly $2nm$ free parameters. On the right-hand side,, there are n free real diagonal parameters in L and $2\frac{n(n-1)}{2} = n^2 - n$ free real off-diagonal one; this makes in total n^2 free parameters in L. Regarding Q, there are a priori $2m$ free parameters in each row, but the first row should have unit norm, the second row should also have unit norm *and* be orthogonal to the first one, and so on. This makes in total

$$(2m-1) + (2m-3) + \ldots + (2(m-n)+1) = m^2 - (m-n)^2 = 2mn - n^2 \quad \text{free real parameters in } Q$$

So the number of free real parameters coincide on each side. The joint distribution of L and Q may therefore be written as

$$p_{L,Q}(L,Q) = p_H(LQ)\,|J_1(L,Q)|$$

where $J_1(L,Q)$ is the Jacobian of the transformation $H \mapsto (L,Q)$. The computation of this Jacobian, that we shall skip here, gives

$$J_1(L,Q) = \prod_{j=1}^{n} l_{jj}^{2(m-j)+1}$$

Besides,

$$p_H(LQ) = \frac{1}{\pi^{nm}} \exp\left(-\mathrm{Tr}\left(LQQ^*L^*\right)\right) = \frac{1}{\pi^{nm}} \exp\left(-\mathrm{Tr}\left(LL^*\right)\right)$$

so finally, we obtain

$$P_{L,Q}(L,Q) = \frac{1}{\pi^{nm}} \exp\left(-\mathrm{Tr}\left(LL^*\right)\right) \prod_{j=1}^{n} l_{jj}^{2(m-j)+1} \, 1_{l_{jj}\geq 0}$$

As this expression does not depend explicitly on Q, this says two things: 1) the distribution of Q is uniform over the set of $n \times m$ complex matrices such that $QQ^* = I_n$ (we will come back to this below); 2) L and Q are actually independent! Let us now look more closely at the distribution of L:

$$\begin{aligned}
p_L(L) &= c_{n,m} \exp\left(-\mathrm{Tr}\left(LL^*\right)\right) \prod_{j=1}^{n} l_{jj}^{2(m-j)+1} \, 1_{l_{jj}\geq 0} \\
&= c_{n,m} \prod_{j=1}^{n} \left(l_{jj}^{2(m-j)+1} \exp\left(-l_{jj}^2\right) \, 1_{l_{jj}\geq 0} \right) \prod_{k<j} \exp\left(-|l_{jk}|^2\right)
\end{aligned}$$

Notice that $c_{n,m}$ above is not equal to $1/\pi^{nm}$, nor is the normalization constant in the uniform distribution $p_Q(Q)$ equal to 1 (the latter is actually equal to $1/V_{n,m}$, where $V_{n,m}$ is the volume of the set of $n \times m$ complex matrices Q such that $QQ^* = I_n$). What the above equality tells us is that:

1) all the entries of L are independent;

2) the off-diagonal entries l_{jk} are i.i.d.$\sim \mathbb{N}_{\mathbb{C}}(0,1)$ random variables;

3) the diagonal entry l_{jj} is a $\chi_{2(m-j+1)}$ random variable.

Remark. Even though we do not need it in the following, let us just make clear what we mean by "uniform" while talking about the distribution of Q. For every fixed $m \times m$ unitary matrix $U \in U(m)$, the matrices H and HU share the same distribution, so the same holds for LQ and LQU. By the independence of L and Q and the non-singularity of the distribution of L, this is saying that Q and QU share the same distribution, for every fixed $m \times m$ unitary matrix U. It is actually a fact that there is only one such distribution, that we call the uniform distribution over the set of $n \times m$ complex matrices Q such that $QQ^* = I_n$.

Joint distribution of the entries of W. We now consider the change of variables $L \mapsto W = LL^*$. Let us compute the number of free real parameters on each side. We have seen above that L contains n^2 free real parameters. The same holds for W, as W contains n diagonal free real parameters and $2\frac{n(n-1)}{2} = n^2 - n$ off-diagonal free real parameters (remembering that W is Hermitian). Considering the reverse transformation $W \mapsto L$ (just because it is easier), we obtain

$$p_L(L) = p_W\left(LL^*\right) \, |J_2(L)|$$

where $J_2(L)$ is the Jacobian of the transformation $W \mapsto L$. The computation of this Jacobian, that we shall again skip here, gives:

$$J_2(L) = 2^n \prod_{j=1}^{n} l_{jj}^{2(n-j)+1}$$

We therefore deduce that

$$p_W(LL^*) = \frac{P_L(L)}{|J_2(L)|} = c_{n,m} \exp\left(-\mathrm{Tr}\left(LL^*\right)\right) \prod_{j=1}^{n} l_{jj}^{2(m-n)}$$

where the constant $c_{n,m}$ differs from the previous one by a factor 2^n, but we keep the same notation for simplicity. Noticing that $LL^* = W$ and $\prod_{j=1}^n l_{jj}^2 = |\det(L)|^2 = \det(LL^*) = \det(W)$, we finally obtain the joint distribution of the entries of W:

$$p_W(W) = c_{n,m} \det(W)^{m-n} \exp(-\mathrm{Tr}(W))\, 1_{\{W \geq 0\}}$$

Remark. In the case $n = m$, the above distribution reads

$$p_W(W) = c_{n,n} \exp(-\mathrm{Tr}(W))\, 1_{\{W \geq 0\}}$$

which looks like a particularly simple expression: it indeed says on one hand that the diagonal entries w_{jj} are i.i.d. exponential random variables; on the other hand, the condition $W \geq 0$ induces lots of subtle constraints and dependencies between the off-diagonal entries.

1.2 Real case

The corresponding model in the real case is given by $W = HH^T$, where H is an $n \times m$ random matrix with i.i.d.$\sim \mathcal{N}_{\mathbb{R}}(0,1)$ entries. Without repeating the whole reasoning, let us briefly mention the corresponding result in this case. We again assume that $m \geq n$ without loss of generality. In this case, the joint distribution of the entries of H is given by

$$p_H(H) = \frac{1}{(2\pi)^{nm/2}} \exp\left(\frac{1}{2}\,\mathrm{Tr}\left(HH^T\right)\right)$$

and the joint distribution of the entries of W is given by

$$p_W(W) = c_{n,m} \det(W)^{\frac{m-n-1}{2}} \exp\left(-\frac{1}{2}\,\mathrm{Tr}(W)\right) 1_{\{W \geq 0\}}$$