Random matrices and communication systems: WEEK 14

1 Diversity-multiplexing tradeoff (cont'd)

Remember from last lecture that we are after computing the diversity order of a multiple antenna channel:

$$d(r) = \lim_{P\to\infty} -\frac{\log(P_{\rm out}(r\,\log P))}{\log P} = \lim_{P\to\infty} -\frac{\log(\mathbb{P}(\log\det(I+PHH^*) < r\,\log P)))}{\log P}$$

where H is an $n \times n$ matrix with i.i.d.$\sim \mathcal{N}_{\mathbb{C}}(0,1)$ entries (and n is fixed). The above probability can be rewritten as

$$\mathbb{P}(\log\det(I+PHH^*) < r\,\log P) = \mathbb{P}\left(\sum_{j=1}^n \log(1+P\lambda_j) < r\,\log P\right)$$

where $\lambda_1,\ldots,\lambda_n$ are the eigenvalues of the matrix HH^*, which are all non-negative. In Lecture 6, we have seen that the joint distribution of these eigenvalues is given by

$$p(\lambda_1,\ldots,\lambda_n) = c_n \prod_{j=1}^n e^{-\lambda_j} \prod_{j<k} (\lambda_k-\lambda_j)^2 \quad \text{for } \lambda_1,\ldots,\lambda_n \geq 0$$

where c_n is some positive constant. Using this, the above probability can be further rewritten as an n-fold integral:

$$\mathbb{P}(\log\det(I+PHH^*) < r\,\log P) = \int_{D_\lambda(r)} p(\lambda_1,\ldots,\lambda_n)\, d\lambda_1\cdots d\lambda_n$$

where

$$D_\lambda(r) = \left\{ 0 \leq \lambda_1 \leq \ldots \leq \lambda_n \ : \ \sum_{j=1}^n \log(1+P\lambda_j) < r\,\log P \right\}$$

(notice that the eigenvalues $\lambda_1,\ldots,\lambda_n$ are ordered in increasing order here). The explicit computation of this integral remains a challenge, because of the highly correlated nature of the eigenvalues. We will see below that taking the high SNR limit ($P\to\infty$) in the above expression allows to drastically simplify the analysis. To this end, let us make the change of variables:

$$\lambda_j = P^{-\alpha_j} = \exp(-\alpha_j\,\log P), \quad \text{so} \quad d\lambda_j = -(\log P)\,\exp(-\alpha_j\,\log P)\,d\alpha_j$$

This change of variable, even though depending on P, is perfectly valid for given value of P, and therefore also in the limit $P\to\infty$ (provided some care is taken here). This gives rise to the following expression for the above probability:

$$\mathbb{P}(\log\det(I+PHH^*) < r\,\log P) = \int_{D_\alpha(r)} q(\alpha_1,\ldots,\alpha_n)\, d\alpha_1\cdots d\alpha_n$$

where

$$q(\alpha_1,\ldots,\alpha_n) = c_n\,\exp\left(-\sum_{j=1}^n P^{-\alpha_j}\right) \prod_{j<k} \left(P^{-\alpha_j} - P^{-\alpha_k}\right)^2 (\log P)^n\,\exp\left(-\sum_{j=1}^n \alpha_j\,\log P\right)$$

and

$$D_\alpha(r) = \left\{ \alpha_1 \geq \ldots \geq \alpha_n\ (\alpha_j \in \mathbb{R}) \ : \ \sum_{j=1}^n \log\left(1+P^{1-\alpha_j}\right) < r\,\log P \right\}$$

So far, these are exact expressions. We will now make a series of approximations which are valid in the limit $P\to\infty$ (and which can all be rigorously justified by taking upper and lower bounds).

First observe that

$$\exp\left(-P^{-\alpha_j}\right) \begin{cases} \text{decays super-polynomially to zero} & \text{if } \alpha_j < 0 \\ \text{tends to 1} & \text{if } \alpha_j \geq 0 \end{cases}$$

so we may restrict the domain of integration $D_\alpha(r)$ to its positive part where $\alpha_1 \geq \ldots \geq \alpha_n \geq 0$.

Next, observe that

$$\log\left(1 + P^{1-\alpha_j}\right) \simeq \begin{cases} (1-\alpha_j)\,\log P & \text{if } \alpha_j \leq 1 \\ 0 & \text{if } \alpha_j > 1 \end{cases}$$

so $\log\left(1 + P^{1-\alpha_j}\right) \simeq (1-\alpha_j)^+ \log P$. We can therefore replace the domain of integration $D_\alpha(r)$ by

$$\widetilde{D}_\alpha(r) = \left\{ \alpha_1 \geq \ldots \geq \alpha_n \geq 0 \;:\; \sum_{j=1}^n (1-\alpha_j)^+ \leq r \right\}$$

and the above probability can be rewritten as

$$\mathbb{P}(\log\det(I + PHH^*) < r\,\log P) \doteq \int_{\widetilde{D}_\alpha(r)} \widetilde{q}(\alpha_1, \ldots, \alpha_n)\, d\alpha_1 \cdots d\alpha_n$$

where

$$\widetilde{q}(\alpha_1, \ldots, \alpha_n) = c_n \prod_{j<k} \left(P^{-\alpha_j} - P^{-\alpha_k}\right)^2 (\log P)^n \exp\left(-\sum_{j=1}^n \alpha_j \,\log P\right)$$

Furthermore, let us notice that $c_n(\log P)^n \doteq 1$, as $\lim_{P\to\infty} \frac{\log(c_n\,(\log P)^n)}{\log P} = 0$. Here comes now the "magic" trick: for $\alpha_1 > \ldots > \alpha_n$, we have

$$\prod_{j<k} \left(P^{-\alpha_j} - P^{-\alpha_k}\right)^2 \doteq \prod_{j<k} P^{-2\alpha_k} = \prod_{k=1}^n P^{-2(k-1)\,\alpha_k} = \exp\left(-\sum_{k=1}^n 2(k-1)\,\alpha_k\,\log P\right)$$

This implies that

$$\widetilde{q}(\alpha_1, \ldots, \alpha_n) \doteq \exp\left(-\sum_{j=1}^n (2j-1)\,\alpha_j\,\log P\right)$$

In this expression, we see that in the limit $P \to \infty$, the exponents α_j become so to speak "independent". Finally, we obtain

$$\mathbb{P}(\log\det(I + PHH^*) < r\,\log P) \doteq \int_{\widetilde{D}_\alpha(r)} \exp\left(-\sum_{j=1}^n (2j-1)\,\alpha_j\,\log P\right) d\alpha_1 \cdots d\alpha_n$$

This expression can in turn be rewritten as

$$\mathbb{P}(\log\det(I + PHH^*) < r\,\log P) \doteq \int_{\widetilde{D}_\alpha(r)} P^{-f(\alpha)}\, d\alpha_1 \cdots d\alpha_n$$

where

$$f(\alpha) = \sum_{j=1}^n (2j-1)\,\alpha_j$$

Using then Laplace's integration method, we obtain

$$\mathbb{P}(\log\det(I + PHH^*) < r\,\log P) \doteq P^{-d(r)}$$

where the diversity order $d(r)$ is given by

$$d(r) = \inf_{\widetilde{D}_\alpha(r)} f(\alpha) = \inf_{\substack{\alpha_1 \geq \ldots \geq \alpha_n \geq 0 \\ \sum_{j=1}^n (1-\alpha_j)^+ < r}} \sum_{j=1}^n (2j-1)\, \alpha_j$$

Doing this, we have therefore transformed the initial problem of evaluating an n-fold integral (in the limit $P \to \infty$) into a simple linear optimization probelem.

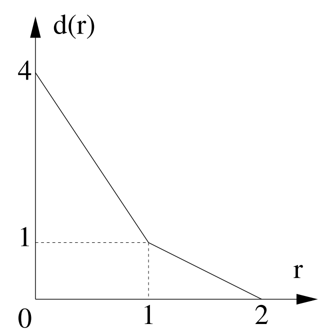
For $n = 2$, the problem reads

$$d(r) = \inf_{\substack{\alpha_1 \geq \alpha_2 \geq 0 \\ (1-\alpha_1)^+ + (1-\alpha_2)^+ \leq r}} \alpha_1 + 3\,\alpha_2$$

whose solution is given by

$$\begin{cases} 0 \leq r \leq 1: & \alpha_1 = 1,\ \alpha_2 = 1 - r,\ d(r) = 4 - 3r \\ 1 \leq r \leq 2: & \alpha_1 = 2 - r,\ \alpha_2 = 0,\ d(r) = 2 - r \end{cases}$$

For low multiplexing gain ($0 \leq r \leq 1$), outage occurs when both eigenvalues λ_1, λ_2 of HH^* are small (more precisely, $\lambda_1 \simeq P^{-1}$ and $\lambda_2 \simeq P^{r-1}$), while for higher multiplexing gain ($1 \leq r \leq 2$), outage occurs when only the smallest eigenvalue λ_2 is small (more precisely $\lambda_1 \simeq 1$ and $\lambda_2 \simeq P^{r-2}$). As expected, the diversity drops to zero for values of r larger than or equal to 2 (as in this case, the target rate is higher than the ergodic capacity). On the figure below, the diversity order is drawn as a function of the multiplexing gain r, which illustrates the tradefoff between diversity and multiplexing.



For general values of n, the curve $d(r)$ is the piecewise linear curve such that $d(k) = (n-k)^2$ at integer values of r (so $d(0) = n^2$ and $d(n) = 0$). Notice that the maximum diversity $d = n^2$ corresponding to $r = 0$ matches the number of independent random variables in the channel matrix H.