1 Diversity-multiplexing tradeoff (cont’d)

Remember from last lecture that we are after computing the diversity order of a multiple antenna channel:

$$d(r) = \lim_{P \to \infty} \frac{-\log(P_{\text{out}}(r \log P))}{\log P} = \lim_{P \to \infty} \frac{-\log(\mathbb{P}(\log \det(I + PHH^*) < r \log P))}{\log P}$$

where \( H \) is an \( n \times n \) matrix with i.i.d. \( \mathcal{N}_C(0, 1) \) entries (and \( n \) is fixed). The above probability can be rewritten as

$$\mathbb{P}(\log \det(I + PHH^*) < r \log P) = \mathbb{P} \left( \sum_{j=1}^{n} \log(1 + P\lambda_j) < r \log P \right)$$

where \( \lambda_1, \ldots, \lambda_n \) are the eigenvalues of the matrix \( HH^* \), which are all non-negative. In Lecture 6, we have seen that the joint distribution of these eigenvalues is given by

$$p(\lambda_1, \ldots, \lambda_n) = c_n \prod_{j=1}^{n} e^{-\lambda_j} \prod_{j<k} (\lambda_k - \lambda_j)^2 \quad \text{for} \quad \lambda_1, \ldots, \lambda_n \geq 0$$

where \( c_n \) is some positive constant. Using this, the above probability can be further rewritten as an \( n \)-fold integral:

$$\mathbb{P}(\log \det(I + PHH^*) < r \log P) = \int_{D_\lambda(r)} p(\lambda_1, \ldots, \lambda_n) d\lambda_1 \cdots d\lambda_n$$

where

$$D_\lambda(r) = \left\{ 0 \leq \lambda_1 \leq \ldots \leq \lambda_n : \sum_{j=1}^{n} \log(1 + P\lambda_j) < r \log P \right\}$$

(notice that the eigenvalues \( \lambda_1, \ldots, \lambda_n \) are ordered in increasing order here). The explicit computation of this integral remains a challenge, because of the highly correlated nature of the eigenvalues. We will see below that taking the high SNR limit \( (P \to \infty) \) in the above expression allows to drastically simplify the analysis. To this end, let us make the change of variables:

$$\lambda_j = P^{-\alpha_j} = \exp(-\alpha_j \log P), \quad \text{so} \quad d\lambda_j = -(\log P) \exp(-\alpha_j \log P) d\alpha_j$$

This change of variable, even though depending on \( P \), is perfectly valid for given value of \( P \), and therefore also in the limit \( P \to \infty \) (provided some care is taken here). This gives rise to the following expression for the above probability:

$$\mathbb{P}(\log \det(I + PHH^*) < r \log P) = \int_{D_\alpha(r)} q(\alpha_1, \ldots, \alpha_n) d\alpha_1 \cdots d\alpha_n$$

where

$$q(\alpha_1, \ldots, \alpha_n) = c_n \exp \left( -\sum_{j=1}^{n} P^{-\alpha_j} \right) \prod_{j<k} (P^{-\alpha_j} - P^{-\alpha_k})^2 (\log P)^n \exp \left( -\sum_{j=1}^{n} \alpha_j \log P \right)$$

and

$$D_\alpha(r) = \left\{ \alpha_1 \geq \ldots \geq \alpha_n (\alpha_j \in \mathbb{R}) : \sum_{j=1}^{n} \log(1 + P^{1-\alpha_j}) < r \log P \right\}$$

So far, these are exact expressions. We will now make a series of approximations which are valid in the limit \( P \to \infty \) (and which can all be rigorously justified by taking upper and lower bounds).
First observe that
\[
\exp (-P^{-\alpha_j}) \begin{cases} 
\text{decays super-polynomially to zero} & \text{if } \alpha_j < 0 \\
\text{tends to 1} & \text{if } \alpha_j \geq 0 
\end{cases}
\]
so we may restrict the domain of integration \(D_\alpha(r)\) to its positive part where \(\alpha_1 \geq \ldots \geq \alpha_n \geq 0\).

Next, observe that
\[
\log(1 + P^{1-\alpha_j}) \simeq \begin{cases} 
(1 - \alpha_j) \log P & \text{if } \alpha_j \leq 1 \\
0 & \text{if } \alpha_j > 1 
\end{cases}
\]
so \(\log(1 + P^{1-\alpha_j}) \simeq (1 - \alpha_j)^+ \log P\). We can therefore replace the domain of integration \(D_\alpha(r)\) by
\[
\tilde{D}_\alpha(r) = \left\{ \alpha_1 \geq \ldots \geq \alpha_n \geq 0 : \sum_{j=1}^n (1 - \alpha_j)^+ \leq r \right\}
\]
and the above probability can be rewritten as
\[
\mathbb{P}(\log \det(I + PHH^*) < r \log P) \doteq \int_{\tilde{D}_\alpha(r)} \bar{q}(\alpha_1, \ldots, \alpha_n) \, d\alpha_1 \cdots d\alpha_n
\]
where
\[
\bar{q}(\alpha_1, \ldots, \alpha_n) = c_n \prod_{j<k} (P^{-\alpha_j} - P^{-\alpha_k})^2 (\log P)^n \exp \left( -\sum_{j=1}^n \alpha_j \log P \right)
\]
Furthermore, let us notice that \(c_n (\log P)^n \doteq 1\), as \(\lim_{P \to \infty} \frac{\log(c_n (\log P)^n)}{\log P} = 0\). Here comes now the "magic" trick: for \(\alpha_1 > \ldots > \alpha_n\), we have
\[
\prod_{j<k} (P^{-\alpha_j} - P^{-\alpha_k})^2 = \prod_{j<k} P^{-2\alpha_k} = \prod_{k=1}^n P^{-2(k-1)\alpha_k} = \exp \left( -\sum_{k=1}^n 2(k-1)\alpha_k \log P \right)
\]
This implies that
\[
\bar{q}(\alpha_1, \ldots, \alpha_n) \doteq \exp \left( -\sum_{j=1}^n (2j-1)\alpha_j \log P \right)
\]
In this expression, we see that in the limit \(P \to \infty\), the exponents \(\alpha_j\) become so to speak "independent". Finally, we obtain
\[
\mathbb{P}(\log \det(I + PHH^*) < r \log P) \doteq \int_{\tilde{D}_\alpha(r)} \exp \left( -\sum_{j=1}^n (2j-1)\alpha_j \log P \right) \, d\alpha_1 \cdots d\alpha_n
\]
This expression can in turn be rewritten as
\[
\mathbb{P}(\log \det(I + PHH^*) < r \log P) \doteq \int_{\tilde{D}_\alpha(r)} P^{-f(\alpha)} \, d\alpha_1 \cdots d\alpha_n
\]
where
\[
f(\alpha) = \sum_{j=1}^n (2j-1)\alpha_j
\]
Using then Laplace's integration method, we obtain
\[
\mathbb{P}(\log \det(I + PHH^*) < r \log P) \doteq P^{-d(r)}
\]
where the diversity order $d(r)$ is given by

$$d(r) = \inf_{D_n(r)} f(\alpha) = \inf_{\alpha_1 \geq \ldots \geq \alpha_n \geq 0} \sum_{j=1}^{n} (2j - 1) \alpha_j$$

Doing this, we have therefore transformed the initial problem of evaluating an $n$-fold integral (in the limit $P \to \infty$) into a simple linear optimization problem.

For $n = 2$, the problem reads

$$d(r) = \inf_{\alpha_1 \geq \alpha_2 \geq 0} \alpha_1 + 3 \alpha_2 \quad \text{subject to} \quad (1-\alpha_1)^+ + (1-\alpha_2)^+ \leq r$$

whose solution is given by

$$\begin{align*}
0 \leq r \leq 1 : & \quad \alpha_1 = 1, \alpha_2 = 1 - r, \quad d(r) = 4 - 3r \\
1 \leq r \leq 2 : & \quad \alpha_1 = 2 - r, \alpha_2 = 0, \quad d(r) = 2 - r
\end{align*}$$

For low multiplexing gain ($0 \leq r \leq 1$), outage occurs when both eigenvalues $\lambda_1, \lambda_2$ of $HH^*$ are small (more precisely, $\lambda_1 \approx P^{-1}$ and $\lambda_2 \approx P^{r-1}$), while for higher multiplexing gain ($1 \leq r \leq 2$), outage occurs when only the smallest eigenvalue $\lambda_2$ is small (more precisely $\lambda_1 \approx 1$ and $\lambda_2 \approx P^{r-2}$). As expected, the diversity drops to zero for values of $r$ larger than or equal to 2 (as in this case, the target rate is higher than the ergodic capacity). On the figure below, the diversity order is drawn as a function of the multiplexing gain $r$, which illustrates the tradeoff between diversity and multiplexing.

For general values of $n$, the curve $d(r)$ is the piecewise linear curve such that $d(k) = (n - k)^2$ at integer values of $r$ (so $d(0) = n^2$ and $d(n) = 0$). Notice that the maximum diversity $d = n^2$ corresponding to $r = 0$ matches the number of independent random variables in the channel matrix $H$. 