Random matrices and communication systems: WEEK 14

1 Diversity-multiplexing tradeoff (cont’d)

Remember from last lecture that we are after computing the diversity order of a multiple antenna channel:

\[ d(r) = \lim_{P \to \infty} -\frac{\log(P_{\text{out}}(r \log P))}{\log P} = \lim_{P \to \infty} -\frac{\log(P(\log \det(I + PHH^*) < r \log P))}{\log P} \]

where \( H \) is an \( n \times n \) matrix with i.i.d. \( \mathcal{N}_C(0,1) \) entries (and \( n \) is fixed). The above probability can be rewritten as

\[ P(\log \det(I + PHH^*) < r \log P) = P\left(\sum_{j=1}^{n} \log(1 + P\lambda_j) < r \log P\right) \]

where \( \lambda_1, \ldots, \lambda_n \) are the eigenvalues of the matrix \( HH^* \), which are all non-negative. In Lecture 6, we have seen that the joint distribution of these eigenvalues is given by

\[ p(\lambda_1, \ldots, \lambda_n) = c_n \prod_{j=1}^{n} e^{-\lambda_j} \prod_{j<k} (\lambda_k - \lambda_j)^2 \quad \text{for} \quad \lambda_1, \ldots, \lambda_n \geq 0 \]

where \( c_n \) is some positive constant. Using this, the above probability can be further rewritten as an \( n \)-fold integral:

\[ P(\log \det(I + PHH^*) < r \log P) = \int_{D_{\lambda}(r)} p(\lambda_1, \ldots, \lambda_n) d\lambda_1 \cdots d\lambda_n \]

where

\[ D_{\lambda}(r) = \left\{ 0 \leq \lambda_1 \leq \ldots \leq \lambda_n : \sum_{j=1}^{n} \log(1 + P\lambda_j) < r \log P \right\} \]

(notice that the eigenvalues \( \lambda_1, \ldots, \lambda_n \) are ordered in increasing order here). The explicit computation of this integral remains a challenge, because of the highly correlated nature of the eigenvalues. We will see below that taking the high SNR limit (\( P \to \infty \)) in the above expression allows to drastically simplify the analysis. To this end, let us make the change of variables:

\[ \lambda_j = P^{-\alpha_j} = \exp(-\alpha_j \log P), \quad \text{so} \quad d\lambda_j = -\alpha_j \exp(-\alpha_j \log P) \, d\alpha_j \]

This change of variable, even though depending on \( P \), is perfectly valid for given value of \( P \), and therefore also in the limit \( P \to \infty \) (provided some care is taken here). This gives rise to the following expression for the above probability:

\[ P(\log \det(I + PHH^*) < r \log P) = \int_{D_{\alpha}(r)} q(\alpha_1, \ldots, \alpha_n) d\alpha_1 \cdots d\alpha_n \]

where

\[ q(\alpha_1, \ldots, \alpha_n) = c_n \exp\left( -\sum_{j=1}^{n} P^{-\alpha_j} \right) \prod_{j<k} (P^{-\alpha_j} - P^{-\alpha_k})^2 \left( \log P \right)^n \exp\left( -\sum_{j=1}^{n} \alpha_j \log P \right) \]

and

\[ D_{\alpha}(r) = \left\{ \alpha_1 \geq \ldots \geq \alpha_n (\alpha_j \in \mathbb{R}) : \sum_{j=1}^{n} \log\left(1 + P^{1-\alpha_j}\right) < r \log P \right\} \]

So far, these are exact expressions. We will now make a series of approximations which are valid in the limit \( P \to \infty \) (and which can all be rigorously justified by taking upper and lower bounds).
First observe that
\[ \exp(-P^{-\alpha}) \begin{cases} \text{decays super-polynomially to zero} & \text{if } \alpha_j < 0 \\ \text{tends to 1} & \text{if } \alpha_j \geq 0 \end{cases} \]
so we may restrict the domain of integration \( D_\alpha \) to its positive part where \( \alpha_1 \geq \ldots \geq \alpha_n \geq 0 \).

Next, observe that
\[ \log(1 + P^{1-\alpha}) \simeq \begin{cases} (1 - \alpha) \log P & \text{if } \alpha_j \leq 1 \\ 0 & \text{if } \alpha_j > 1 \end{cases} \]
so \( \log(1 + P^{1-\alpha}) \simeq (1 - \alpha_j)^+ \log P \). We can therefore replace the domain of integration \( D_\alpha \) by
\[ \tilde{D}_\alpha(r) = \left\{ \alpha_1 \geq \ldots \geq \alpha_n \geq 0 : \sum_{j=1}^n (1 - \alpha_j)^+ \leq r \right\} \]
and the above probability can be rewritten as
\[ \mathbb{P}(\log \det(I + P H H^*) < r \log P) \doteq \int_{\tilde{D}_\alpha(r)} \tilde{q}(\alpha_1, \ldots, \alpha_n) \, d\alpha_1 \cdots d\alpha_n \]
where
\[ \tilde{q}(\alpha_1, \ldots, \alpha_n) = c_n \prod_{j<k} (P^{-\alpha_j} - P^{-\alpha_k})^2 (\log P)^n \exp \left( -\sum_{j=1}^n \alpha_j \log P \right) \]
Furthermore, let us notice that \( c_n (\log P)^n \doteq 1 \), as \( \lim_{P \to \infty} \frac{\log(c_n (\log P)^n)}{\log P} = 0 \). Here comes now the "magic" trick: for \( \alpha_1 > \ldots > \alpha_n \), we have
\[ \prod_{j<k} (P^{-\alpha_j} - P^{-\alpha_k})^2 \simeq \prod_{j<k} P^{-2 \alpha_k} = \prod_{k=1}^n P^{-2(k-1)\alpha_k} = \exp \left( -\sum_{k=1}^n 2(k-1) \alpha_k \log P \right) \]
This implies that
\[ \tilde{q}(\alpha_1, \ldots, \alpha_n) \doteq \exp \left( -\sum_{j=1}^n (2j - 1) \alpha_j \log P \right) \]
In this expression, we see that in the limit \( P \to \infty \), the exponents \( \alpha_j \) become so to speak "independent". Finally, we obtain
\[ \mathbb{P}(\log \det(I + P H H^*) < r \log P) \doteq \int_{\tilde{D}_\alpha(r)} \exp \left( -\sum_{j=1}^n (2j - 1) \alpha_j \log P \right) \, d\alpha_1 \cdots d\alpha_n \]
This expression can in turn be rewritten as
\[ \mathbb{P}(\log \det(I + P H H^*) < r \log P) \doteq \int_{\tilde{D}_\alpha(r)} P^{-f(\alpha)} \, d\alpha_1 \cdots d\alpha_n \]
where
\[ f(\alpha) = \sum_{j=1}^n (2j - 1) \alpha_j \]
Using then Laplace’s integration method, we obtain
\[ \mathbb{P}(\log \det(I + P H H^*) < r \log P) \doteq P^{-d(r)} \]
where the diversity order $d(r)$ is given by

$$d(r) = \inf_{D_{\alpha}(r)} \inf_{\alpha_1 \geq \alpha_2 \geq 0} \sum_{j=1}^{n-1} (2j-1) \alpha_j$$

Doing this, we have therefore transformed the initial problem of evaluating an $n$-fold integral (in the limit $P \to \infty$) into a simple linear optimization problem.

For $n = 2$, the problem reads

$$d(r) = \inf_{\alpha_1 \geq 1, \alpha_2 \geq 0} \alpha_1 + 3 \alpha_2$$

whose solution is given by

$$\begin{cases}
0 \leq r \leq 1 : & \alpha_1 = 1, \alpha_2 = 1 - r, d(r) = 4 - 3r \\
1 \leq r \leq 2 : & \alpha_1 = 2 - r, \alpha_2 = 0, d(r) = 2 - r
\end{cases}$$

For low multiplexing gain ($0 \leq r \leq 1$), outage occurs when both eigenvalues $\lambda_1, \lambda_2$ of $HH^*$ are small (more precisely, $\lambda_1 \simeq P^{-1}$ and $\lambda_2 \simeq P^{r-1}$), while for higher multiplexing gain ($1 \leq r \leq 2$), outage occurs when only the smallest eigenvalue $\lambda_2$ is small (more precisely $\lambda_1 \simeq 1$ and $\lambda_2 \simeq P^{r-2}$). As expected, the diversity drops to zero for values of $r$ larger than or equal to 2 (as in this case, the target rate is higher than the ergodic capacity). On the figure below, the diversity order is drawn as a function of the multiplexing gain $r$, which illustrates the tradeoff between diversity and multiplexing.

For general values of $n$, the curve $d(r)$ is the piecewise linear curve such that $d(k) = (n - k)^2$ at integer values of $r$ (so $d(0) = n^2$ and $d(n) = 0$). Notice that the maximum diversity $d = n^2$ corresponding to $r = 0$ matches the number of independent random variables in the channel matrix $H$. 