Random matrices and communication systems: WEEK 13

1 Capacity of multiple antenna channels

1.1 Finite-size anaylsis

Let us come back the multiple antenna channel considered in Lecture 3:

$$Y = H\,X + Z$$

where H is an $n \times n$ random channel matrix with i.i.d. $\mathcal{N}_{\mathbb{C}}(0,1)$ entries, varying ergodically over time, whose realizations are known at the receiver, but not at the transmitter. We have seen in Lecture 3 that the ergodic capacity of such a system is given by

$$C_{\text{erg}} = \mathbb{E}\left(\log\det\left(I + \frac{P}{n}\,HH^*\right)\right) = \mathbb{E}\left(\sum_{j=1}^{n} \log\left(1 + P\lambda_j^{(n)}\right)\right)$$

where the expectation is taken over all possible realizations of the random matrix H and $\lambda_1^{(n)}, \ldots, \lambda_n^{(n)}$ are the non-negative eigenvalues of the $n \times n$ Wishart matrix $W^{(n)} = \frac{1}{n}\,HH^*$. This may be further rewritten as

$$C_{\text{erg}} = n\,\mathbb{E}\left(\log\left(1 + P\lambda^{(n)}\right)\right)$$

where $\lambda^{(n)}$ is one of the eigenvalues $\lambda_1^{(n)}, \ldots, \lambda_n^{(n)}$ picked uniformly at random. We have seen in Lecture 7 that the distribution of $\lambda^{(n)}$ is given by

$$p^{(n)}(\lambda) = e^{-n\lambda} \sum_{l=0}^{n-1} L_l(n\lambda)^2$$

where the $L_l(\cdot)$ are the Laguerre polynomials. Therefore,

$$C_{\text{erg}} = n \int_0^\infty d\lambda\, p^{(n)}(\lambda)\, \log(1 + P\lambda) = n \sum_{l=0}^{n-1} \int_0^\infty d\lambda\, e^{-n\lambda}\, L_l(n\lambda)^2\, \log(1 + P\lambda)$$

1.2 Asymptotic analysis

In order to analyze the behavior of the ergodic capacity in the large n limit, let us rewrite it as

$$C_{\text{erg}}(n) = \mathbb{E}\left(\sum_{j=1}^{n} \log\left(1 + P\lambda_j^{(n)}\right)\right) = n\,\mathbb{E}\left(\int_{\mathbb{R}} \log(1 + Px)\, d\mu_n(x)\right)$$

where

$$\mu_n = \frac{1}{n} \sum_{j=1}^{n} \delta_{\lambda_j^{(n)}}$$

is the empirical eigenvalue distribution of the matrix $W^{(n)} = \frac{1}{n}\,HH^*$. We have seen in Lectures 10-12 that almost surely, μ_n converges weakly towards the limiting deterministic distribution μ whose pdf is given by

$$p_\mu(x) = \frac{1}{\pi}\sqrt{\frac{1}{x} - \frac{1}{4}}\, 1_{\{0<x<4\}}$$

This is saying that almost surely, for any bounded continuous function $f : \mathbb{R} \to \mathbb{R}$,

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(x)\, d\mu_n(x) = \int_{\mathbb{R}} f(x) d\mu(x)$$

Taking expectations on both sides (which is OK here, thanks to the dominated convergence theorem), we obtain

$$\lim_{n\to\infty} \mathbb{E}\left(\int_{\mathbb{R}} f(x)\, d\mu_n(x)\right) = \int_{\mathbb{R}} f(x) d\mu(x)$$

for any bounded continuous function $f : \mathbb{R} \to \mathbb{R}$ (remember that μ is deterministic). We would like now to apply this to $f(x) = \log(1 + Px)$, which would allow us to conclude that

$$\lim_{n\infty} \frac{C_{\text{erg}}(n)}{n} = \lim_{n\to\infty} \mathbb{E}\left(\int_{\mathbb{R}} \log(1 + Px)\, d\mu_n(x)\right) = \int_0^4 \log(1 + Px)\, p_\mu(x)\, dx$$

therefore proving that $C_{\text{erg}}(n)$ is of order n for large values of n and at the same time providing an explicit expression for the multiplicative factor.

A first concern is that $f(x) = \log(1 + Px)$ is not defined for $x < -1/P$, but as both μ_n and μ are supported on $[0, \infty)$, this is not a problem. The other worry is that f is unbounded. To this end, let us define $f_M(x) = \min(f(x), M)$, which is bounded and continuous for any $M > 0$. This allows us to write, for fixed values of n and M

$$\begin{aligned}
\left|\mathbb{E}\left(\int_0^\infty f(x)\, d\mu_n(x)\right) - \int_0^\infty f(x) d\mu(x)\right| &\le \left|\mathbb{E}\left(\int_0^\infty f(x)\, d\mu_n(x)\right) - \mathbb{E}\left(\int_0^\infty f_M(x)\, d\mu_n(x)\right)\right| \\
&+ \left|\mathbb{E}\left(\int_0^\infty f_M(x)\, d\mu_n(x)\right) - \int_0^\infty f_M(x)\, d\mu(x)\right| + \left|\int_0^\infty f_M(x)\, d\mu(x) - \int_0^\infty f_(x)\, d\mu(x)\right| \\
&\le \ \mathbb{E}\left(\int_{x_M}^\infty f(x)\, d\mu_n(x)\right) + \left|\mathbb{E}\left(\int_0^\infty f_M(x)\, d\mu_n(x)\right) - \int_0^\infty f_M(x)\, d\mu(x)\right| + \int_{x_M}^\infty f(x)\, d\mu(x)
\end{aligned}$$

where $x_M = \inf\{x > 0 \,:\, f(x) \ge M\} = \frac{1}{P}\,(e^M - 1)$. By the weak convergence result above, we know that the term in the middle converges to zero for any value of M, so

$$\lim_{n\to\infty} \left|\mathbb{E}\left(\int_0^\infty f(x)\, d\mu_n(x)\right) - \int_0^\infty f(x) d\mu(x)\right| \le \lim_{n\to\infty} \mathbb{E}\left(\int_{x_M}^\infty f(x)\, d\mu_n(x)\right) + \int_{x_M}^\infty f(x)\, d\mu(x)$$

for any $M > 0$. We also know that $\int_{x_M}^\infty f(x)\, d\mu(x) = 0$ for $x_M > 4$ (as μ is supported on $[0, 4]$), so there remains to prove that (also for $x_M > 4$)

$$\lim_{n\to\infty} \mathbb{E}\left(\int_{x_M}^\infty f(x)\, d\mu_n(x)\right) = 0$$

Notice that

$$\mathbb{E}\left(\int_{x_M}^\infty f(x)\, d\mu_n(x)\right) = \frac{1}{n}\sum_{j=1}^n \mathbb{E}\left(f(\lambda_j^{(n)})\, 1_{\{\lambda_j^{(n)} \ge x_M\}}\right) \le \mathbb{E}\left(f(\lambda_{\max}^{(n)})\, 1_{\{\lambda_{\max}^{(n)} \ge x_M\}}\right)$$

As $f(x) = \log(1 + Px) \le Px$, this is further bounded above by

$$P\, \mathbb{E}\left(\lambda_{\max}^{(n)}\, 1_{\{\lambda_{\max}^{(n)} \ge x_M\}}\right)$$

In Lecture 11, we have seen (under slightly different assumptions, though) that $\lim_{n\to\infty} \mathbb{E}(\lambda_{\max}^{(n)}) \le 4$. Similar refined estimates on $\lambda_{\max}^{(n)}$ allow to conclude that the above expression converges to zero as $n \to \infty$ for $x_M > 4$.

2 Diversity-multiplexing tradeoff

Consider now the same scenario as above, except for the fact that H is fixed over time (see Lecture 4). In this case, the capacity of the multiple antenna channel is equal to zero, and the outage probability is given by

$$\mathbb{P}_{\text{out}}(R) = \inf_{Q \geq 0 \,:\, \text{Tr}(Q) \leq P} \mathbb{P}(\log\det(I + HQH^*) < R)$$

where again, the probability is taken over all possible realizations of the random matrix H. For a fixed value of n, we would like to characterize the behavior of this outage probability in the high SNR regime (that is, as P gets large), with the idea that it can be made vanishingly small in this regime. In the ergodic case, we have seen that for large P (see Lecture 3),

$$C_{\text{erg}} = \sup_{Q \geq 0 \,:\, \text{Tr}(Q) \leq P} \mathbb{E}(\log\det(I + HQH^*)) \simeq n \log P$$

So in order to obtain a small outage probability, the target rate R chosen in the above expression should not be higher than $n \log P$. Let us therefore choose $R = r \log P$, where $0 \leq r \leq n$: r is called the target *multiplexing gain.*

A priori, the analysis of the outage probability is particularly difficult, as the above minimization problem remains unsolved. But notice that

$$\mathbb{P}(\log\det(I + PHH^*) < r \log P) \leq \mathbb{P}_{\text{out}}(r \log P) \leq \mathbb{P}\left(\log\det\left(I + \frac{P}{n} HH^*\right) < r \log P\right)$$

Indeed, $Q = \frac{P}{n} I$ is a possible candidate for the minimization problem, which explains the inequality on the right-hand side. On the other hand, any matrix $Q \geq 0$ satisfying $\text{Tr}(Q) \leq P$ also satisfies $Q \leq PI$, which implies the inequality on the left-hand side. Observe now that as $P \to \infty$,

$$\begin{aligned} &\mathbb{P}\left(\log\det\left(I + \frac{P}{n} HH^*\right) < r \log P\right) \doteq \mathbb{P}(\log\det(I + PHH^*) < r \log(nP)) \\ &= \mathbb{P}(\log\det(I + PHH^*) < r (\log n + \log P)) \doteq \mathbb{P}(\log\det(I + PHH^*) < r \log P) \end{aligned}$$

where the notation $f(P) \doteq g(P)$ stands for $\lim_{P\to\infty} \frac{\log(f(P))}{\log P} = \lim_{P\to\infty} \frac{\log(g(P))}{\log P}$. This together with the previous inequalities allows us to conclude that

$$\mathbb{P}_{\text{out}}(r \log P) \doteq \mathbb{P}(\log\det(I + PHH^*) < r \log P)$$

As mentioned above, by choosing the multiplexing gain r smaller than n, we expect the outage probability to converge to zero as P gets large. Our aim in the following is to discover at which speed, depending on r, does this probability converge to zero, namely to find the exponent $d(r)$ satisfying

$$P_{\text{out}}(r \log P) \doteq P^{-d(r)}$$

More formally, this exponent, also known as the *diversity order*, is defined as

$$d(r) = \lim_{P\to\infty} -\frac{\log(P_{\text{out}}(r \log P))}{\log P}$$

which the above analysis allows us to rewrite as

$$d(r) = \lim_{P\to\infty} -\frac{\log(\mathbb{P}(\log\det(I + PHH^*) < r \log P))}{\log P}$$

The computation of $d(r)$, which requires the knowledge of the joint eigenvalue distribution of the matrix HH^*, will be the subject of the next lecture.