Random matrices and communication systems: WEEK 12

In this lecture, we reprove the theorem from last time using the Stieltjes transform method.

1 Marčenko-Pastur’s theorem

Let $H$ be an $n \times n$ random matrix with i.i.d. complex-valued entries such that for all $1 \leq j, l \leq n$:

(i) $E(h_{jl}) = 0$, $E(|h_{jl}|^2) = 1$;

(ii) the distribution of $h_{jl}$ is compactly supported (this second assumption may be relaxed).

As last time, let us consider $W(n) = \frac{1}{n} HH^*$, $\lambda_1^{(n)}, \ldots, \lambda_n^{(n)}$ its (non-negative) eigenvalues and the empirical distribution $\mu_n = \frac{1}{n} \sum_{j=1}^n \delta_{\lambda_j^{(n)}}$. A particular instance of Marčenko-Pastur’s theorem is the following.

**Theorem 1.1.** Under assumptions (i) and (ii), almost surely, the sequence $(\mu_n, n \geq 1)$ converges weakly towards the quarter-circle law $\mu$, whose pdf is given by

$$p_\mu(x) = \frac{1}{\pi} \sqrt{\frac{1}{x} - \frac{1}{4}} 1_{\{0 < x < 4\}}$$

As a preliminary to the proof of the theorem, let us consider, for a given $z \in \mathbb{C} \setminus \mathbb{R}$, the quadratic equation:

$$z g(z)^2 + z g(z) + 1 = 0 \quad (1)$$

This quadratic equation has two solutions

$$g_\pm(z) = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{1}{z}}$$

It turns out that $g_+(z)$ is the Stieltjes transform of the above distribution $\mu$. The proof is left as an exercise in the homework.

*Proof of Theorem 1.1 (sketch).*

The strategy for today is to use the characterization of weak convergence via Stieltjes transform: a sequence of distributions converges weakly towards a limiting distribution if the corresponding sequence of Stieltjes transforms converges pointwise on $\mathbb{C}_+ = \{ z \in \mathbb{C} : \text{Im } z > 0 \}$ towards the limiting Stieltjes transform. Thus, we will prove that almost surely, $g_n(z)$ converges in the large $n$ limit towards a solution of equation (1). As all the $g_n$ are by definition Stieltjes transforms, but only one of the two solutions of this equation is a Stieltjes transform, a continuity argument allows then to conclude that $g_n$ can only converge to $g_+$ and not to $g_-$.

The rule of the game is therefore now to try writing $g_n(z)$ on both sides of an equality sign. To this end, let us compute

$$g_n(z) = \int_{\mathbb{R}} \frac{1}{x-z} \, d\mu_n(x) = \frac{1}{n} \sum_{j=1}^n \frac{1}{\lambda_j^{(n)} - z} = \frac{1}{n} \text{Tr} \left( \left( W(n) - z I \right)^{-1} \right)$$

First notice that $W(n) = \frac{1}{n} HH^* = \frac{1}{n} \sum_{k=1}^n h_k h_k^*$, where $h_k$ is the $k^{th}$ column of $H$ (and is therefore $n \times 1$). This way, $W(n)$ is expressed as a sum of rank one $n \times n$ matrices. For a given $1 \leq k \leq n$, let us define

$$W_k^{(n)} = W(n) - \frac{1}{n} h_k h_k^*$$

as well as the resolvents

$$G^{(n)}(z) = \left( W(n) - z I \right)^{-1} \quad \text{and} \quad G_k^{(n)}(z) = \left( W_k^{(n)} - z I \right)^{-1}$$

First notice that $W_k^{(n)} = \frac{1}{n} \sum_{l=1, l \neq k}^n h_l h_l^*$.
Notice that the object we are interested in is \( g_n(z) = \frac{1}{n} \text{Tr} \left( G_n(z) \right) \). Let us now prove the following two lemmas.

**Lemma 1.2.**

\[
\frac{1}{n} h_k^* G_n(z) h_k = \frac{1}{n} h_k^* G_k (z) h_k + \frac{1}{n} G_k (z) h_k
\]

**Proof.** Let us compute

\[
h_k^* G_k (z) \left( G_k (z) \right)^{-1} = h_k^* G_k (z) \left( W_k (z) - zI \right) = h_k^* G_k (z) \left( W_k (z) - zI + \frac{1}{n} h_k h_k^* \right)
\]

which implies by the Borel-Cantelli lemma that

\[
\frac{1}{n} h_k^* G_k (z) h_k
\]

Therefore,

\[
h_k^* G_k (z) = \left( 1 + \frac{1}{n} G_k (z) h_k h_k^* \right) h_k^* G_k (z)
\]

and

\[
h_k^* G_k (z) h_k = \left( 1 + \frac{1}{n} G_k (z) h_k h_k^* \right) h_k^* G_k (z) h_k
\]

which concludes the proof. \( \square \)

**Lemma 1.3.**

\[g_n(z) = \frac{1}{n} \text{Tr} \left( G_n(z) \right) = - \frac{1}{n} \sum_{k=1}^{n} \frac{1}{1 + \frac{1}{n} h_k^* G_k (z) h_k}\]

**Proof.** Let us compute

\[
1 = \frac{1}{n} \text{Tr}(I) = \frac{1}{n} \text{Tr} \left( \left( W_n(i) - zI \right) \right) G_n(z) = \frac{1}{n} \text{Tr} \left( \frac{1}{n} \sum_{k=1}^{n} h_k h_k^* G_k (z) - z G_k (z) \right)
\]

\[
= \frac{1}{n} \sum_{k=1}^{n} \left( \frac{1}{n} h_k^* G_k (z) h_k \right) - z g_n(z) = \frac{1}{n} \sum_{k=1}^{n} \left( \frac{1}{n} h_k^* G_k (z) h_k \right) - z g_n(z)
\]

by Lemma 1.2. So

\[
z g_n(z) = \frac{1}{n} \sum_{k=1}^{n} \left( \frac{1}{n} h_k^* G_k (z) h_k - 1 \right) = - \frac{1}{n} \sum_{k=1}^{n} \frac{1}{1 + \frac{1}{n} h_k^* G_k (z) h_k}
\]

which concludes the proof. \( \square \)

Notice that so far, these formulas hold for any matrix of the form \( W_n = \frac{1}{n} H H^* \), without any further assumption on the matrix \( H \). On the contrary, the next lemma relies strongly on the assumptions (i) and (ii).

**Lemma 1.4.** Under assumptions (i) and (ii), for all \( z \in \mathbb{C} \setminus \mathbb{R} \) and all \( \varepsilon > 0 \), there exists \( C > 0 \) independent of \( n \) such that

\[
\mathbb{P} \left( \left| \frac{1}{n} h_k^* G_k (z) h_k - \frac{1}{n} \text{Tr} \left( G_k (z) \right) \right| \geq \varepsilon \right) \leq \frac{C}{n^2}
\]

which implies by the Borel-Cantelli lemma that

\[
\frac{1}{n} h_k^* G_k (z) h_k - \frac{1}{n} \text{Tr} \left( G_k (z) \right) \xrightarrow{n \to \infty} 0 \quad \text{almost surely}
\]
We do not prove this lemma, but simply show in the following that for all \( n \geq 1 \),

\[
E \left( \frac{1}{n} h_k^* G_k^{(n)} h_k - \frac{1}{n} \text{Tr} \left( G_k^{(n)}(z) \right) \right) = 0
\]

which also requires the use of the assumptions made above:

\[
E \left( \frac{1}{n} h_k^* G_k^{(n)} h_k \right) = \frac{1}{n} \sum_{j,l=1}^{n} \mathbb{E} \left( \overline{h_{jk}} \left( W_k^{(n)} - zI \right)^{-1}_{jl} h_{lk} \right)
\]

As the matrix \( W_k^{(n)} \) does not “contain” \( h_k \), the \( k^{th} \) column of \( H \), it is independent of both \( \overline{h_{jk}} \) and \( h_{lk} \), so

\[
E \left( \frac{1}{n} h_k^* G_k^{(n)} h_k \right) = \frac{1}{n} \sum_{j,l=1}^{n} \mathbb{E} \left( \overline{h_{jk}} \left( W_k^{(n)} - zI \right)^{-1}_{jl} \right) = \mathbb{E} \left( \frac{1}{n} \text{Tr} \left( W_k^{(n)} - zI \right) \right) = \mathbb{E} \left( \frac{1}{n} \text{Tr} \left( G_k^{(n)}(z) \right) \right)
\]

where \((*)\) follows from the fact that \( E(\overline{h_{jk}} h_{lk}) = \delta_{jl} \), according to assumption (i) and the independence assumption.

The actual proof of Lemma 1.4 relies on the use of Chebychev’s inequality with \( \phi(x) = x^4 \) and a similar analysis of the expectation (except that one must consider the moment of order 4 instead of the first order moment).

The last lemma is a technicality, which holds again for any matrix of the form \( W^{(n)} = \frac{1}{n} HH^* \).

**Lemma 1.5.** For all \( z \in \mathbb{C} \setminus \mathbb{R} \),

\[
\left| \frac{1}{n} \text{Tr} \left( G_k^{(n)}(z) \right) - \frac{1}{n} \text{Tr} \left( G_k^{(n)}(z) \right) \right| \leq \frac{1}{n |\text{Im} \ z|}
\]

The proof of this lemma is still rather long for a technicality and is therefore omitted.

Gathering together the results of Lemmas 1.3, 1.4 and 1.5, we obtain that for large values of \( n \),

\[
g_n(z) = \frac{1}{n} \text{Tr} \left( G_k^{(n)}(z) \right) = -\frac{1}{nz} \sum_{k=1}^{n} \frac{1}{1 + \frac{1}{n} h_k^* G_k^{(n)}(z) h_k}
\]

\[
\simeq -\frac{1}{nz} \sum_{k=1}^{n} \frac{1}{1 + \frac{1}{n} \text{Tr} \left( G_k^{(n)}(z) \right)} \simeq -\frac{1}{z(1 + g_n(z))} = -\frac{1}{z\left(1 + g_n(z)\right)}
\]

which may be rewritten as

\[
z g_n(z)^2 + z g_n(z) + 1 \simeq 0
\]

Taking some more precautions, we can conclude that \( g_n(z) \) converges almost surely towards a solution of the quadratic equation (1), which should be chosen as \( g_+ \) for the reasons explained above. This “completes” the proof of the theorem.