Random matrices and communication systems: WEEK 11

1 End of the proof of Wigner's theorem

Let us first recall equation (6) from last lecture:

$$\mathbb{E}\left(m_k^{(n)}\right) = \frac{1}{n^{k+1}} \sum_{j_1, l_1, \dots, j_k, l_k=1}^n \mathbb{E}\left(h_{j_1, l_1} \overline{h_{j_2, l_1}} \cdots h_{j_k, l_k} \overline{h_{j_1, l_k}}\right)$$
(1)

Our aim in the following is to deduce from there that

$$\left|\mathbb{E}\left(m_{k}^{(n)}\right) - c_{k}\right| = O\left(\frac{1}{n}\right) \tag{2}$$

where c_k is the k^{th} Catalan number. For a given $k \ge 0$, we need to find out which of the terms in (1) bring a non-negligible contribution to this expression in the large n limit. Observe first that to each term

$$\mathbb{E}\left(h_{j_1,l_1}\,\overline{h_{j_2,l_1}}\cdots h_{j_k,l_k}\,\overline{h_{j_1,l_k}}\right)$$

corresponds a sequence $(j_1, l_1, j_2, l_2, ..., j_k, l_k)$, or equivalently a directed bipartite graph from $\{1, ..., n\}$ to $\{1, ..., n\}$, defined pictorially as follows:



Figure 1. Directed bipartite graph associated to $\mathbb{E}\left(h_{j_1,l_1}\overline{h_{j_2,l_1}}h_{j_2,l_2}\overline{h_{j_3,l_2}}h_{j_3,l_3}\overline{h_{j_1,l_3}}\right)$

The vertex j_1 is called the *root* of the graph (it is both its starting and ending point). We will say that two sequences or graphs have the same *structure* if the order of appearance of new vertices in the sequence is the same. For example, the two sequences on the left-hand side below have the same structure, while the two on the right-hand side don't:

j_1	l_1	j_2	l_2	j_3	l_3	j_4	l_4	j_1	l_1	j_2	l_2	j_3	l_3	j_4	l_4
1	7	1	4	3	7	3	2	1	7	1	4	3	7	3	2
2	4	2	5	8	4	8	4	2	4	2	5	8	5	8	4

Because of assumption (iii) on the matrix entries h_{jl} , as well as the independence assumption, we see that in order for a given sequence to bring a non-zero contribution, it is necessary that whenever an edge from j to l appears in the graph (possibly a certain number of times), then it should also appear the *same* number of times in the opposite direction (that is, from l to j). A sequence or graph with this property is said to be *even*. Notice that an even graph with 2k edges can have at most k + 1 vertices, as each edge in the graph is doubled.

The question is now: for an even graph with 2k edges, p vertices and a given structure, how many graphs with the same structure can we possibly have by permuting the positions of the vertices on each side? Clearly, there are at most n choices for each vertex, so the total number of choices is less than n^p .

Therefore,

$$\mathbb{E}\left(m_{k}^{(n)}\right) = \frac{1}{n^{k+1}} \sum_{\substack{(j_{1},l_{1},\ldots,j_{k},l_{k}) \text{ even}}} \mathbb{E}\left(h_{j_{1},l_{1}} \overline{h_{j_{2},l_{1}}} \cdots h_{j_{k},l_{k}} \overline{h_{j_{1},l_{k}}}\right)$$

$$= \frac{1}{n^{k+1}} \sum_{p=2}^{k+1} \sum_{\substack{(j_{1},l_{1},\ldots,j_{k},l_{k}) \text{ even with } p \text{ vertices}}} \mathbb{E}\left(h_{j_{1},l_{1}} \overline{h_{j_{2},l_{1}}} \cdots h_{j_{k},l_{k}} \overline{h_{j_{1},l_{k}}}\right)$$

$$= \frac{1}{n^{k+1}} \sum_{\substack{(j_{1},l_{1},\ldots,j_{k},l_{k}) \text{ even with } k+1 \text{ vertices}}} \mathbb{E}\left(h_{j_{1},l_{1}} \overline{h_{j_{2},l_{1}}} \cdots h_{j_{k},l_{k}} \overline{h_{j_{1},l_{k}}}\right) + O\left(\frac{1}{n}\right)$$

(Observe also that for a given p, each term is the sum is bounded because of assumption (ii).)

In conclusion, the only graphs that can possibly bring a non-negligible contribution are even graphs with p = k + 1 vertices. In such graphs, each edge leads to a new vertex, so the resulting graph is actually a *tree*. For a tree with a given structure, there are order n^{k+1} different choices for the positions of the vertices. To be more precise, as the graph is bipartite, there are

$$n(n-1)\dots(n-p_1+1)n(n-1)\dots(n-p_2+1)$$
 choices (3)

where p_1, p_2 are the number of vertices on both sides of the graph, with $p_1 + p_2 = k + 1$. In all cases, the above expression is of order $n^{p_1+p_2} = n^{k+1}$ as n grows large and k remains fixed. This factor n^{k+1} compensates therefore exactly with the $1/n^{k+1}$ factor in front of the sum. Notice that in such graphs, each edge appears exactly once in each direction of the original directed graph, so $\mathbb{E}\left(|h_{j_1,l_1} \overline{h_{j_2,l_1}} \cdots h_{j_k,l_k} \overline{h_{j_1,l_k}}\right)$ is a product of $\mathbb{E}\left(|h_{jl}|^2\right) = 1$ by assumption (i), therefore the product itself is equal to 1.

The only question remaining is therefore: how many different structures of even graphs with 2k edges and k + 1 vertices do there exist for a given k? In order to answer this question, let us make yet another identification: starting from the root j_1 , explore the corresponding directed graph "following the arrows" and draw next to that a path that goes either up or down by one unit at each time step, according to the following rule:

if the current edge is a new edge, then go up by one unit

if the current edge has already been visited (in the other direction, necessarily), then go down by one unit

The path being drawn is nothing but a Dyck path seen at the beginning of this lecture: it starts in 0, lands in 0 after 2k steps and cannot drop below zero in the meanwhile. Therefore, the number of different possible tree structures with 2k edges is equal to the number Dyck paths of length 2k, that is, the Catalan number c_k . Gathering all the above observations together, we finally obtain (2):

$$\mathbb{E}\left(m_{k}^{(n)}\right) = c_{k} + O\left(\frac{1}{n}\right)$$

2 Largest eigenvalue

The results obtained in the previous lecture tell us something about the asymptotic distribution of a "typical" eigenvalue of the matrix $W^{(n)}$, that is, an eigenvalue picked uniformly at random. What can we say now on the *extreme eigenvalues* of such matrices, that is, the largest and the smallest one $\lambda_{\max}^{(n)}$ and $\lambda_{\min}^{(n)}$? It is first important to remember what Wigner's theorem actually says: the fact that almost surely, μ_n converges weakly towards μ means that for all a < b

$$\mu_n([a,b]) = \frac{1}{n} \, \sharp \{ 1 \le j \le n \; : \; \lambda_j^{(n)} \in [a,b] \} \underset{n \to \infty}{\to} \mu([a,b]) = \int_{a \lor 0}^{b \land 4} p_\mu(x) dx$$

Therefore, as soon as the interval [a, b] has a non-empty intersection with the open interval]0, 4[, the quantity on the right-hand side is strictly positive. This is saying in turn that the number of eigenvalues in this interval grows linearly in n as n grows to infinity. This applies in particular to the intervals $[0, \varepsilon]$ and $[4 - \varepsilon, 4]$, for any fixed $\varepsilon > 0$, implying that almost surely, as n grows to infinity, both

$$\lim_{n \to \infty} \lambda_{\min}^{(n)} \le \varepsilon \quad \text{and} \quad \lim_{n \to \infty} \lambda_{\max}^{(n)} \ge 4 - \varepsilon$$

and therefore

$$\lim_{n \to \infty} \lambda_{\min}^{(n)} \le 0 \quad \text{and} \quad \lim_{n \to \infty} \lambda_{\max}^{(n)} \ge 4$$

In the present case, this settles the limiting value of the smallest eigenvalue, as we know on the other hand that $\lambda_{\min}^{(n)} \geq 0$ for all n, because $W^{(n)}$ is positive semi-definite.

On the contrary, it is unclear whether $\lim_{n\to\infty} \lambda_{\max}^{(n)} = 4$. Indeed, it could well be that one eigenvalue diverges from the interval [0, 4] in the large *n* limit: this would not affect the result of the Wigner theorem. Indeed, the weight of one eigenvalue in the distribution μ_n is equal to 1/n, so an isolated eigenvalue cannot contribute to change the limiting distribution μ .

In order to study the asymptotic behavior of $\lambda_{\max}^{(n)}$, we will again use moments. This is made possible thanks to the following fact: for all $k \ge 1$,

$$\lambda_{\max}^{(n)} = \left((\lambda_{\max}^{(n)})^k \right)^{1/k} \le \left(\sum_{j=1}^n (\lambda_j^{(n)})^k \right)^{1/k} = \left(\operatorname{Tr} \left((W^{(n)})^k \right) \right)^{1/k}$$

 \mathbf{SO}

$$\lambda_{\max}^{(n)} \leq \lim_{k \to \infty} \left(\operatorname{Tr} \left((W^{(n)})^k \right) \right)^{1/k}$$

(and this last bound is actually known to be tight). We now set out to prove that under the following assumption:

$$h_{ik} = \exp(i\phi_{jk})$$
 where ϕ_{jk} are i.i.d. $\sim \mathcal{U}([0, 2\pi])$ random variables (4)

(which implies assumptions (i)-(iii) made in the last lecture), we have

$$\lim_{n \to \infty} \mathbb{E}\left(\lambda_{\max}^{(n)}\right) = 4 \tag{5}$$

Using more refined techniques, one can prove the same result under weaker assumptions, as well as the almost sure version of the result: let us skip this.

Proof of equation (5). From the explanations above, it is clear that $\lim_{n\to\infty} \mathbb{E}\left(\lambda_{\max}^{(n)}\right) \geq 4$, so what remains to be proven is the upper bound

$$\lim_{n \to \infty} \mathbb{E}\left(\lambda_{\max}^{(n)}\right) \le 4$$

Notice first that as $f(x) = x^{1/k}$ is concave, we obtain by Jensen's inequality¹,

$$\mathbb{E}\left(\lambda_{\max}^{(n)}\right) = \mathbb{E}\left(\lim_{k \to \infty} \left(\operatorname{Tr}\left((W^{(n)})^k\right)\right)^{1/k}\right) \le \lim_{k \to \infty} \mathbb{E}\left(\operatorname{Tr}\left((W^{(n)})^k\right)\right)^{1/k}$$

As we have seen before,

$$\mathbb{E}\left(\mathrm{Tr}\left((W^{(n)})^k\right)\right) = \frac{1}{n^k} \mathbb{E}\left(\mathrm{Tr}\left((HH^*)^k\right)\right) = \frac{1}{n^k} \sum_{j_1, l_1, \dots, j_k, l_k=1}^n \mathbb{E}\left(h_{j_1, l_1} \overline{h_{j_2, l_1}} \cdots h_{j_k, l_k} \overline{h_{j_1, l_k}}\right)$$

which is equation (1), up to a factor 1/n. We now perform a similar analysis as before, but notice that we are interested in a different order of limits here: we first take $k \to \infty$ and then $n \to \infty$.

¹and also by Fatou's lemma, which is needed here in order to interchange the limit and the expectation

Observe that as before, only the sequences $(j_1, l_1, \ldots, j_k, l_k)$ that correspond to even bipartite graphs bring a non-zero contribution to the sum, so

$$\mathbb{E}\left(\operatorname{Tr}\left((W^{(n)})^{k}\right)\right) = \frac{1}{n^{k}} \sum_{\substack{(j_{1},l_{1},\ldots,j_{k},l_{k}) \text{ even}}} \mathbb{E}\left(h_{j_{1},l_{1}} \overline{h_{j_{2},l_{1}}} \cdots h_{j_{k},l_{k}} \overline{h_{j_{1},l_{k}}}\right)$$
$$= \frac{1}{n^{k}} \sum_{p=2}^{k+1} \sum_{\substack{(j_{1},l_{1},\ldots,j_{k},l_{k}) \\ \text{even with } p \text{ vertices}}} \mathbb{E}\left(h_{j_{1},l_{1}} \overline{h_{j_{2},l_{1}}} \cdots h_{j_{k},l_{k}} \overline{h_{j_{1},l_{k}}}\right)$$

Notice also that for an even graph, the expression

$$\mathbb{E}\left(h_{j_1,l_1}\,\overline{h_{j_2,l_1}}\cdots h_{j_k,l_k}\,\overline{h_{j_1,l_k}}\right)$$

is the expectation of a product of even powers of $|h_{jl}|$, which are all equal to 1 here, because of assumption (4). So we simply have

$$\mathbb{E}\left(\operatorname{Tr}\left((W^{(n)})^k\right)\right) = \frac{1}{n^k} \sum_{p=2}^{k+1} N(k,p)$$

where N(k, p) denotes the number of even graphs with 2k edges and p vertices (with $2 \le p \le k+1$). We have seen before that

$$N(k,k+1) \sim c_k \, n^{k+1}$$

as n gets large. It also holds that

$$\sum_{p=2}^{k+1} N(k,p) \le c_k \, n^{k+1}$$

Indeed, when counting the number of graphs with 2k edges and k + 1 vertices, we identified above c_k different structures for such graphs and slightly less than n^{k+1} graphs with a given structure, because of the constraint of having disjoint vertices on each side of the graph; see equation (3). Observe now that if we relax this constraint, we obtain all possible graphs with 2k edges and k + 1 or less vertices. As there are at most n choices for each vertex, the total number of such graphs does not exceed $c_k n^{k+1}$.

This finally implies that

$$\mathbb{E}\left(\lambda_{\max}^{(n)}\right) \le \lim_{k \to \infty} \left(\frac{1}{n^k} c_k n^{k+1}\right)^{1/k} = \lim_{k \to \infty} (nc_k)^{1/k} \le 4, \quad \text{for all } n \ge 1$$

as we have already seen that $c_k \leq 4^k$ (and notice that as we consider first $k \to \infty$ and n fixed, the multiplicative factor n disappears in the large k limit). This completes the proof. \Box .