Graded homework 4 (due Monday, May 28, 1:15 PM)

Exercise 1. Let μ be a distribution and $(m_k,\, k \geq 0)$ the the sequence of its moments. For a given $n \geq 1$, let also $A^{(n)}$ be the $(n+1) \times (n+1)$ matrix defined as $a_{jk}^{(n)} = m_{j+k}$, $0 \leq j, k \leq n$.

a) Show that for all $n \geq 1$, the matrix $A^{(n)}$ is positive semi-definite, i.e. that for all $c_0, \ldots, c_n \in \mathbb{R}$,

$$\sum_{j,k=0}^{n} c_j\, c_k\, a_{jk}^{(n)} \geq 0$$

NB: $A^{(n)}$ is of course also symmetric.

b) Among the following sequences of numbers, which are sequences of moments of a given distribution? and of what distribution?

i) $\left(m_k = \dfrac{1}{k+1},\, k \geq 0\right)$ *ii)* $\left(m_k = k^2,\, k \geq 0\right)$ *iii)* $\left(m_k = e^k,\, k \geq 0\right)$ *iv)* $\left(m_k = e^{k^2/2},\, k \geq 0\right)$

Exercise 2. Compute the moments of the following distributions **and** tell which of them are uniquely determined by their moments, using Carleman's condition.

a) Let μ be the "quarter-circle law" whose pdf is given by

$$p_\mu(x) = \frac{1}{\pi} \sqrt{\frac{1}{x} - \frac{1}{4}}\, 1_{\{0<x<4\}}.$$

Hint: use induction and the change of variable $x = 4\sin^2 t$.

b) Let $\lambda > 0$ and μ be the distribution whose pdf is given by

$$p_\mu(x) = C_\lambda \exp(-x^\lambda), \quad x > 0,$$

with C_λ an appropriate normalization constant. For which values of λ is the distribution μ_λ uniquely determined by its moments? (an exact computation of the moments is not required here).

Hint: use the approximation $\Gamma(x) = \int_0^\infty y^{x-1}\, e^{-y}\, dy \sim [x]!$ as $x \to \infty$.

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Exercise 3. Let μ be a probability distribution on $\mathbb{R}$ and $g_\mu : \mathbb{C}\backslash\mathbb{R} \to \mathbb{C}$ be its Stieltjes transform, defined as

$$g_\mu(z) = \int_{\mathbb{R}} \frac{1}{x-z}\, d\mu(x), \quad z \in \mathbb{C}\backslash\mathbb{R}.$$

a) Writing $z = u + iv$, decompose $g_\mu(z)$ into its real and imaginary parts.

b*) Show that g_μ is analytic on $\mathbb{C}\backslash\mathbb{R}$

c) Show that $\text{Im}\ g_\mu(z) > 0$, if $\text{Im}\ z > 0$.

d) Show that $\lim_{v\to+\infty} v\,|g_\mu(iv)| = 1$.

e) Show that $g_\mu(\overline{z}) = \overline{g_\mu(z)}$.

NB: If a function g satisfies properties b) to e), then it is the Stieltjes transform of a distribution μ (BTW, property c) ensures that $\mu(B) \geq 0$ for all $B \in \mathcal{B}(\mathbb{R})$ and property d) ensures that $\mu(\mathbb{R}) = 1$).

Exercise 4. a) Let μ be any probability distribution on $\mathbb{R}$. For any $a < b$ continuity points of F_μ, prove that

$$\frac{1}{\pi} \lim_{\varepsilon\downarrow 0} \int_a^b \text{Im}\ g_\mu(x + i\varepsilon)\, dx = F_\mu(b) - F_\mu(a).$$

b) Assume now that μ admits a pdf p_μ. Show that for any $x \in \mathbb{R}$,

$$p_\mu(x) = \frac{1}{\pi} \lim_{\varepsilon\downarrow 0} \text{Im}\ g_\mu(x + i\varepsilon).$$

c) Application: Let g_μ be the Stieltjes transform solution of the quadratic equation

$$z\, g_\mu(z)^2 + z\, g_\mu(z) + 1 = 0, \quad z \in \mathbb{C}\backslash\mathbb{R}.$$

Deduce what distribution μ corresponds to g_μ.

NB: The above quadratic equation has two solutions, but notice that only one solution is a Stieltjes transform satisfying the properties of Exercise 3.