## Graded homework 3 (due Monday, April 30, 1:15 PM)

**Exercise 1. a) Matrix norms:** For a given $n \times n$ matrix $A = (a_{jk})$, we define

$$\|A\|_1 = \sup_{u \in \mathbb{C}^n,\, u \neq 0} \frac{\|Au\|}{\|u\|} \quad \text{and} \quad \|A\|_2 = \sqrt{\frac{1}{n}\operatorname{Tr}(A^*A)}.$$

Show that

$$\left|\frac{1}{n}\operatorname{Tr}(A)\right| \leq \|A\|_2 \leq \|A\|_1.$$

For two $n \times n$ matrices $A$ and $B$, show moreover that

$$\|AB\|_1 \leq \|A\|_1 \, \|B\|_1 \quad \text{and} \quad \|AB\|_2 \leq \|A\|_1 \, \|B\|_2.$$

**b) Relation with eigenvalues:**
Let $\alpha_1, \ldots, \alpha_n$ be the eigenvalues of $A$ and $\beta_1, \ldots, \beta_n$ be the eigenvalues of $A^*A$.
What do we know *a priori* about the $\beta$'s?

In the case where $A$ is Hermitian (i.e. $A = A^*$), what do we know *a priori* about the $\alpha$'s?
and what is the relation between the $\alpha$'s and the $\beta$'s?

Show that in general, the following holds:

$$\|A\|_1 = \max_{j \in \{1,\ldots,n\}} \sqrt{\beta_j}, \quad \|A\|_2 = \sqrt{\frac{1}{n}\sum_{j=1}^n \beta_j}, \quad \text{and} \quad \frac{1}{n}\operatorname{Tr}(A^m) = \frac{1}{n}\sum_{j=1}^n \alpha_j^m, \quad \text{for any } m \geq 0.$$

**c) Asymptotic equivalence of Toeplitz and ciculant matrices:**
(first step in the proof of the Grenander-Szegö theorem = Lemma 1 of the course)

Let $t_0 = 2$, $t_1 = t_{-1} = -1$ and $t_l = 0$ for $|l| > 1$. Let $(T^{(n)}, n \geq 1)$ be the sequence of Toeplitz matrices built from the sequence $(t_l,\, l \in \mathbb{Z})$. For each $n$, let $C^{(n)}$ be the circulant matrix corresponding to $T^{(n)}$ (that is, the matrix $T^{(n)}$ with 0 replaced by $-1$ in the lower left and upper right corners).

Show that there exists $K > 0$ such that

$$\sup_{n \geq 1} \|T^{(n)}\|_1 \leq K \quad \text{and} \quad \sup_{n \geq 1} \|C^{(n)}\|_1 \leq K \quad \text{and} \quad \|T^{(n)} - C^{(n)}\|_2 \leq \frac{K}{\sqrt{n}}.$$

Noticing that for any $m \geq 0$, (this can be proved easily by induction)

$$(T^{(n)})^m - (C^{(n)})^m = \sum_{j=1}^m (C^{(n)})^{j-1} \, (T^{(n)} - C^{(n)}) \, (T^{(n)})^{m-j},$$

show, using most of the preceding statements, that if $\lambda_k^{(n)}$, $\mu_k^{(n)}$ are the eigenvalues of $T^{(n)}$, $C^{(n)}$, respectively, then for any fixed $m \geq 0$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n \left(\lambda_k^{(n)}\right)^m = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n \left(\mu_k^{(n)}\right)^m.$$

please turn the page

**Exercise 2. Proof of Lemma 2 of the course**
Let $(t_l,\, l \in \mathbb{Z})$ be a sequence of complex numbers such that $t_l = 0$ for all $|l| > l_0$ and $t_{-l} = \overline{t_l}$ for all $|l| \leq l_0$. Let $T^{(n)}$ be the $n \times n$ matrix with entries $T^{(n)}_{jk} = t_{k-j}$ and $g$ be the function defined as

$$g(x) = \sum_{l=-l_0}^{l_0} t_l\, e^{ilx}, \quad x \in [0, 2\pi].$$

Show that if $\lambda^{(n)}$ is an eigenvalue of $T^{(n)}$, then

$$\lambda^{(n)} \in \mathbb{R} \quad \text{and} \quad \inf_{x\in[0,2\pi]} g(x) \leq \lambda^{(n)} \leq \sup_{x\in[0,2\pi]} g(x).$$

*Hint:* compute $u^*(T^{(n)})u$ for the eigenvector $u$ corresponding to $\lambda^{(n)}$ and use the fact that

$$t_l = \frac{1}{2\pi} \int_0^{2\pi} g(x)\, e^{-ilx}\, dx.$$

**Exercise 3. An example**
Let $\rho \in ]-1, +1[$ and $T^{(n)}$ be the $n \times n$ Toeplitz matrix whose entries are given by

$$T^{(n)}_{jk} = \rho^{|j-k|}.$$

Using the Grenander-Szegö theorem, compute *both* the function $g(x)$ and the limiting eigenvalue distribution $p(y)$ of $T^{(n)}$ as $n \to \infty$.