Graded homework 3 (due Monday, April 30, 1:15 PM)

Exercise 1. a) Matrix norms: For a given $n \times n$ matrix $A = (a_{jk})$, we define

$$
\|A\|_1 = \sup_{u \in \mathbb{C}^n, u \neq 0} \frac{\|Au\|}{\|u\|} \quad \text{and} \quad \|A\|_2 = \sqrt{\frac{1}{n} \text{Tr}(A^*A)}.
$$

Show that

$$
\left| \frac{1}{n} \text{Tr}(A) \right| \leq \|A\|_2 \leq \|A\|_1.
$$

For two $n \times n$ matrices $A$ and $B$, show moreover that

$$
\|AB\|_1 \leq \|A\|_1 \|B\|_1 \quad \text{and} \quad \|AB\|_2 \leq \|A\|_1 \|B\|_2.
$$

b) Relation with eigenvalues:

Let $\alpha_1, \ldots, \alpha_n$ be the eigenvalues of $A$ and $\beta_1, \ldots, \beta_n$ be the eigenvalues of $A^*A$.

What do we know a priori about the $\beta$’s?

In the case where $A$ is Hermitian (i.e. $A = A^*$), what do we know a priori about the $\alpha$’s?

and what is the relation between the $\alpha$’s and the $\beta$’s?

Show that in general, the following holds:

$$
\|A\|_1 = \max_{j \in \{1,\ldots,n\}} \sqrt{\beta_j}, \quad \|A\|_2 = \sqrt{\frac{1}{n} \sum_{j=1}^n \beta_j}, \quad \text{and} \quad \frac{1}{n} \text{Tr}(A^m) = \frac{1}{n} \sum_{j=1}^n \alpha_j^m, \quad \text{for any } m \geq 0.
$$

c) Asymptotic equivalence of Toeplitz and circulant matrices:

(first step in the proof of the Grenander-Szegö theorem = Lemma 1 of the course)

Let $t_0 = 2$, $t_1 = t_{-1} = -1$ and $t_l = 0$ for $|l| > 1$. Let $(T^{(n)}, n \geq 1)$ be the sequence of Toeplitz matrices built from the sequence $(t_l, l \in \mathbb{Z})$. For each $n$, let $C^{(n)}$ be the circulant matrix corresponding to $T^{(n)}$ (that is, the matrix $T^{(n)}$ with 0 replaced by $-1$ in the lower left and upper right corners).

Show that there exists $K > 0$ such that

$$
\sup_{n \geq 1} \|T^{(n)}\|_1 \leq K \quad \text{and} \quad \sup_{n \geq 1} \|C^{(n)}\|_1 \leq K \quad \text{and} \quad \|T^{(n)} - C^{(n)}\|_2 \leq \frac{K}{\sqrt{n}}.
$$

Noticing that for any $m \geq 0$, (this can be proved easily by induction)

$$
(T^{(n)})^m - (C^{(n)})^m = \sum_{j=1}^m (C^{(n)})^{j-1} (T^{(n)} - C^{(n)}) (T^{(n)})^{m-j},
$$

show, using most of the preceding statements, that if $\lambda^{(n)}_k$, $\mu^{(n)}_k$ are the eigenvalues of $T^{(n)}$, $C^{(n)}$, respectively, then for any fixed $m \geq 0$,

$$
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n (\lambda^{(n)}_k)^m = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n (\mu^{(n)}_k)^m.
$$

please turn the page
Exercise 2. Proof of Lemma 2 of the course

Let \((t_l, l \in \mathbb{Z})\) be a sequence of complex numbers such that \(t_l = 0\) for all \(|l| > l_0\) and \(t_{-l} = \overline{t}_l\) for all \(|l| \leq l_0\). Let \(T^{(n)}\) be the \(n \times n\) matrix with entries \(T^{(n)}_{jk} = t_{k-j}\) and \(g\) be the function defined as

\[
g(x) = \sum_{l=-l_0}^{l_0} t_l e^{ilx}, \quad x \in [0, 2\pi].
\]

Show that if \(\lambda^{(n)}\) is an eigenvalue of \(T^{(n)}\), then

\[
\lambda^{(n)} \in \mathbb{R} \quad \text{and} \quad \inf_{x \in [0, 2\pi]} g(x) \leq \lambda^{(n)} \leq \sup_{x \in [0, 2\pi]} g(x).
\]

Hint: compute \(u^*(T^{(n)})u\) for the eigenvector \(u\) corresponding to \(\lambda^{(n)}\) and use the fact that

\[
t_l = \frac{1}{2\pi} \int_0^{2\pi} g(x) e^{-ilx} dx.
\]

Exercise 3. An example

Let \(\rho \in ]-1, +1[\) and \(T^{(n)}\) be the \(n \times n\) Toeplitz matrix whose entries are given by

\[
T^{(n)}_{jk} = \rho^{|j-k|}.
\]

Using the Grenander-Szegö theorem, compute both the function \(g(x)\) and the limiting eigenvalue distribution \(p(y)\) of \(T^{(n)}\) as \(n \to \infty\).