Graded Homework 2 (due Thursday, April 5, 1:15 PM)

Exercise 1. Let $H$ be a $2 \times 2$ matrix with i.i.d.$\sim \mathcal{N}_\mathbb{R}(0,1)$ entries and $W = HH^T$.

a) Write down the joint distribution of the entries of $H$.

$H = LQ$, where $L$ is a $2 \times 2$ lower-triangular matrix and $Q$ is a $2 \times 2$ orthogonal matrix (i.e. $QQ^T = I$).

b) Compute the Jacobian of the transformation $H \mapsto (L, Q)$ and deduce from there the joint distribution of $L$ and $Q$ (that is, the joint distribution of $a, b, c, u$ below).

NB: We may assume that $L$ and $Q$ are of the form

$$L = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}, \quad Q = \begin{pmatrix} \cos(u) & \sin(u) \\ -\sin(u) & \cos(u) \end{pmatrix}$$

where $a, c \geq 0$, $b \in \mathbb{R}$ and $u \in [0, 2\pi]$.

From the LQ decomposition, we obtain $W = HH^T = LL^T$.

c) Compute the Jacobian of the transformation $L \mapsto W$ and deduce from there the joint distribution of the entries of $W$.

Exercise 2. Let $H$ be an $n \times n$ complex matrix with i.i.d.$\sim \mathcal{N}_\mathbb{C}(0,1)$ entries and $Q$ be an $n \times n$ deterministic and positive definite matrix. The goal of this exercise is to determine the joint distribution of the eigenvalues of the $n \times n$ matrix $W = HQH^*$ (using the results already mentioned during the course).

a) Show that $W$ is positive semi-definite.

b) Let $M = \text{diag}(\mu_1, \ldots, \mu_n)$, where $\mu_1, \ldots, \mu_n$ are the (positive) eigenvalues of $Q$. Show that $W$ and $HMH^*$ have the same distribution.

c) Compute the joint distribution of the entries of $\tilde{H} = HM^{1/2}$.

NB: $M^{1/2} = \text{diag}(\sqrt{\mu_1}, \ldots, \sqrt{\mu_n})$.

d) Compute the the joint distribution of the entries of the matrix $\tilde{W} = \tilde{H}^*\tilde{H}$.

NB: this is not a typo; we do not consider here $\tilde{H}\tilde{H}^*$.

e) Compute the joint eigenvalue distribution of $\tilde{W}$ (which is the same as that of $W$).

NB: do not worry if you cannot get a completely closed form expression...