Graded Homework 2 (due Thursday, April 5, 1:15 PM)

Exercise 1. Let H be a 2×2 matrix with i.i.d.$\sim \mathcal{N}_{\mathbb{R}}(0,1)$ entries and $W = HH^T$.

a) Write down the joint distribution of the entries of H.

$H = LQ$, where L is a 2×2 lower-triangular matrix and Q is a 2×2 orthogonal matrix (i.e. $QQ^T = I$).

b) Compute the Jacobian of the transformation $H \mapsto (L, Q)$ and deduce from there the joint distribution of L and Q (that is, the joint distribution of a, b, c, u below).

NB: We may assume that L and Q are of the form

$$L = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \quad Q = \begin{pmatrix} \cos(u) & \sin(u) \\ -\sin(u) & \cos(u) \end{pmatrix}$$

where $a, c \geq 0$, $b \in \mathbb{R}$ and $u \in [0, 2\pi]$.

From the LQ decomposition, we obtain $W = HH^T = LL^T$.

c) Compute the Jacobian of the transformation $L \mapsto W$ and deduce from there the joint distribution of the entries of W.

Exercsie 2. Let H be an $n \times n$ complex matrix with i.i.d.$\sim \mathcal{N}_{\mathbb{C}}(0,1)$ entries and Q be an $n \times n$ deterministic and positive definite matrix. The goal of this exercise is to determine the joint distribution of the eigenvalues of the $n \times n$ matrix $W = HQH^*$ (using the results already mentioned during the course).

a) Show that W is positive semi-definitte.

b) Let $M = \mathrm{diag}(\mu_1, \ldots, \mu_n)$, where $\mu_1, \ldots, \mu_n$ are the (positive) eigenvalues of Q. Show that W and HMH^* have the same distribution.

c) Compute the joint distribution of the entries of $\widetilde{H} = HM^{1/2}$.

NB: $M^{1/2} = \mathrm{diag}(\sqrt{\mu_1}, \ldots, \sqrt{\mu_n})$.

d) Compute the the joint distribution of the entries of the matrix $\widetilde{W} = \widetilde{H}^* \widetilde{H}$.

NB: this is *not* a typo; we do not consider here $\widetilde{H}\widetilde{H}^*$.

e) Compute the joint eigenvalue distribution of $\widetilde{W}$ (which is the same as that of W).

NB: do not worry if you cannot get a completely closed form expression...